

VECTOR SPACES

Linear combination:

Let V be a vector space over a field F and S be a non-empty subset of V . Then an element $v \in V$ of the form $d_1 v_1 + d_2 v_2 + \dots + d_n v_n$ where $d_1, d_2, \dots, d_n$ are in F , is called a linear combination of vectors $v_1, v_2, \dots, v_n$.

$$(i.e) \quad v = d_1 v_1 + d_2 v_2 + \dots + d_n v_n$$

Formula $v = a e_1 + b e_2 + c e_3$ (*)

⑦. write the $v = (1, -2, 5)$ as a linear combination of the vectors $e_1 = (1, 1, 1)$, $e_2 = (1, 2, 3)$, $e_3 = (2, -1, 1)$.

Sol

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix} = 1(2+3) - 1(1-6) + 1(-1-4) \\ = 1(5) - 1(-5) + 1(-5) \\ = 5 + 5 - 5$$

$$\Delta = 5 \neq 0$$

It is a linearly independent

It is a basis.

$$(1, -2, 5) = a(1, 1, 1) + b(1, 2, 3) + c(2, -1, 1)$$

$$\Delta_a = \begin{vmatrix} 1 & -2 & 5 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix} = 1(2+3) + 2(1-6) + 5(1-6) \\ = 1(5) + 2(-5) + 5(-5) \\ = 5 - 10 - 25$$

$$\Delta_a = -30$$

$$\Delta_b = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & 5 \\ 2 & -1 & 1 \end{vmatrix} = 1(-2+5) - 1(1-10) + 1(-1+4) \\ = 1(3) - 1(-9) + 1(3) \\ = 3 + 9 + 3 = 15$$

$$\Delta_b = 15$$

$$\Delta c = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -2 & 5 \end{vmatrix} = 1(10-6) - 1(5-3) + 1(-2-2)$$

$$= 1(16) - 1(2) + 1(-4)$$

$$= 16 - 2 - 4$$

$$\Delta c = 10$$

$$a = \frac{\Delta a}{\Delta} = \frac{-30}{5} = -6$$

$$b = \frac{\Delta b}{\Delta} = \frac{15}{5} = 3$$

$$c = \frac{\Delta c}{\Delta} = \frac{10}{5} = 2$$

Solution

$$a = -6$$

$$b = 3$$

$$c = 2$$

From (1).

$$V = ae_1 + be_2 + ce_3$$

$$= -6e_1 + 3e_2 + 2e_3$$

$\therefore V$ is a linear combination of e_1, e_2, e_3 .

② write the vector $V = (2, -5, 3)$ as a linear combination of the vectors $e_1 = (1, -3, 2)$, $e_2 = (2, -1, -1)$, $e_3 = (1, -5, 7)$.

Sol

$$\Delta = \begin{vmatrix} 1 & -3 & 2 \\ 2 & -1 & -1 \\ 1 & -5 & 7 \end{vmatrix} = 1(-28-5) + 2(14+1) + 2(-10+7)$$

$$= 1(-33) + 2(15) + 2(-6)$$

$$= -33 + 45 - 12$$

$$= 45 - 45$$

$$|\Delta| = 0$$

It is a linearly dependent

It is a basis.

It is no solution

V is a not linear combination of e_1, e_2, e_3 .

③ Let $u_1 = (1, 2)$, $u_2 = (2, 1)$ verify whether $v = (2, 2)$ is a linear combination of u_1, u_2 .

Sol

$$\Delta = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1 - 4 = -3 \neq 0$$

$$\Delta \neq 0$$

$$\Delta = -3$$

$\therefore$ It is a linearly independent
It is a basis.

Formula

$$x = a u_1 + b u_2$$

$$(2, 2) = a(1, 2) + b(2, 1)$$

$$\Delta a = \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} = 2 - 4 = -2$$

$$\Delta a = -2$$

$$\Delta b = \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = 2 - 4 = -2$$

$$\Delta b = -2$$

$$a = \frac{\Delta a}{\Delta} = \frac{-2}{-3} = 2/3 \Rightarrow a = 2/3$$

$$b = \frac{\Delta b}{\Delta} = \frac{-2}{-3} = 2/3 \Rightarrow b = 2/3$$

From (1)

$$x = a u_1 + b u_2$$

$$x = \frac{2}{3} u_1 + \frac{2}{3} u_2$$

$\therefore$ It is a linear combination of u_1 & u_2 .

④. Let $u_1 = (2, 1)$, $u_2 = (-1, -2)$ verify whether $x = (3, 5)$
is a linear combination of u_1 & u_2 .

Sol

$$|A| = \begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix} = -4 + 1 = 0$$

$$|A| = 0$$

It is a linear dependent

It is not basis.

⑤. Verify $3x^3 - 2x^2 + 7x + 8$ is a linear combination of $x^3 - 2x^2 - 5x - 3$ and $3x^3 - 5x^2 - 4x - 9$ in $P_3(\mathbb{R})$.

Sol

Formula:

$$\text{Let } Y = aV_1 + bV_2$$

$$3x^3 - 2x^2 + 7x + 8 = a(x^3 - 2x^2 - 5x - 3) + b(3x^3 - 5x^2 - 4x - 9)$$

equating $x^3, x^2, x, \text{ constant}$.

$$a + 3b = 3 \rightarrow (1)$$

$$-2a - 5b = -2 \rightarrow (2)$$

$$-5a - 4b = 7 \rightarrow (3)$$

$$-3a - 9b = 8 \rightarrow (4)$$

$$\textcircled{1} \times 2 \Rightarrow \begin{array}{r} 2a + 6b = 6 \\ -2a - 5b = -2 \\ \hline \end{array}$$

$$\boxed{b = 1}$$

$b = 1$ sub in $\textcircled{1}$

$$a + 3(1) = 3$$

$$a = 3 - 3$$

$$\boxed{a = -9}$$

From $\textcircled{3}$

$$a = -9, b = 1$$

$$-5(-9) - 4(1) = 7$$

$$\begin{array}{r} 45 \\ -4 \\ \hline 41 \end{array} \quad 45 - 4 = 41$$

$$\begin{array}{r} 45 \\ -36 \\ \hline 9 \end{array} \quad 45 - 36 = 9$$

$$\begin{array}{r} 45 \\ -36 \\ \hline 9 \end{array} \quad 45 - 36 = 9$$

From $\textcircled{4}$

$$-3(-9) - 9(1) = 8$$

$$27 - 9 = 18$$

$$18 \neq 8$$

$\therefore \textcircled{3}$ & $\textcircled{4}$ equation are not satisfied.

$\therefore$ It is a not linear combination

$\therefore p(x)$ is not a linear combination of $q(x)$ & $r(x)$.

6. verify $x^3 - 2x^2 + 4x$ is a linear combination of $x^3 - 2x^2 + 3x - 1$ and $x^3 - 2x + 3$.

sol $V = av_1 + bv_2$

$$x^3 - 2x^2 + 4x = a(x^3 - 2x^2 + 3x - 1) + b(x^3 - 2x + 3)$$

$$a + b = 1 \rightarrow (1)$$

$$-2a - 0 = -8 \rightarrow (2)$$

$$3a - 2b = 4 \rightarrow (3)$$

$$-a + 3b = 0 \rightarrow (4)$$

from (2) $-2a = -8$
 $a = +8/2$

$$\boxed{a = +4}$$

sub in (1) $a + b = 1$

$$+4 + b = 1$$

$$b = 1 - 4$$

$$\boxed{b = -3}$$

from (3) $3a - 2b = 4$

$$12 + 6 = 4$$

$$18 \neq 4$$

from (4)

$$-4 - 9 = 0$$

$$-13 = 0$$

$$-13 \neq 0$$

Equations (3) & (4) are not satisfied.

It is not linear combination.

7. consider the vectors $u = (1, 2, -1)$ & $v = (6, 4, 2)$ in $\mathbb{R}^3$. Show that $w = (9, 2, 7)$ is a linear combination of u & v .

sol Given $u = (1, 2, -1)$, $v = (6, 4, 2)$, $w = (9, 2, 7)$

$$w = au + bv$$

$$(9, 2, 7) = a(1, 2, -1) + b(6, 4, 2)$$

$$a + 6b = 9 \rightarrow (1)$$

$$2a + 4b = 2 \rightarrow (2)$$

$$-a + 2b = 7 \rightarrow (3)$$

from (1) & (2)

$$(1) \times 2 \Rightarrow 2a + 12b = 18$$

$$\Rightarrow 2a + 4b = 2$$

$$\boxed{b = 2}$$

$$\text{sub } b = 2 \text{ in } ①$$

$$2a + 4b = 2$$

$$2a + 8 = 2$$

$$2a = 2 - 8$$

$$2a = -6$$

$$a = -3$$

From ①

$$a + b = 9$$

$$-3 + b = 9$$

$$-3 + 12 = 9$$

$$9 = 9$$

From ②

$$2a + 4b = 2$$

$$2(-3) + 4(2) = 2$$

$$-6 + 8 = 2$$

$$2 = 2$$

From ③

$$-a + 2b = 7$$

$$-(-3) + 2(2) = 7$$

$$3 + 4 = 7$$

$$7 = 7$$

$a = -3, b = 2$ satisfies all the equations

$\therefore w$ is a linear combination of u & v .

②. Consider the vectors $u = (1, 2, 3)$ & $v = (2, 3, 1)$ in $\mathbb{R}^3$. For what value k , is the vector $w = (1, k, 4)$ a linear combination of u & v .

Given $w = (1, k, 4), u = (1, 2, 3), v = (2, 3, 1)$

Formula $w = au + bv$

$$(1, k, 4) = a(1, 2, 3) + b(2, 3, 1)$$

$$a + 2b = 1 \rightarrow ①$$

$$2a + 3b = k \rightarrow ②$$

$$3a + b = 4 \rightarrow ③$$

$$① \times 2 \Rightarrow 2a + 4b = 2$$

$$③ \quad \begin{array}{r} 2a + 4b = 2 \\ 3a + b = 4 \\ \hline 5b = -1 \end{array}$$

$$b = -1/5$$

$b = -1/5$ sub in ①

$$a + 2b = 1$$

$$a + 2(-1/5) = 1$$

$$a - 2/5 = 1$$

$$a = 1 + 2/5$$

$$a = 7/5$$

sub $a = 7/5, b = -1/5$ in ②

$$2a + 3b = k$$

$$2(7/5) + 3(-1/5) = k$$

$$14/5 - 3/5 = k$$

$$11/5 = k$$

$$k = 11/5$$

SPAN OF A SET

Let V be vector space over a field F . Let S be a non-empty subset of a vector space V . Then the set of all linear combination of finite sets of elements of S is called a linear span of S , denoted by LCS .

$$d_1 v_1 + d_2 v_2 + d_3 v_3 + \dots + d_n v_n, \quad d_1, d_2, \dots, d_n \in F$$

① show that the vectors $u_1 = (1, 1, 1)$, $u_2 = (1, 2, 3)$ & $u_3 = (1, 5, 8)$ span $\mathbb{R}^3$.

sol

$$u_1 = (1, 1, 1)$$

$$u_2 = (1, 2, 3)$$

$$u_3 = (1, 5, 8)$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 8 \end{vmatrix}$$

$$= 1(16 - 15) - 1(8 - 7) + 1(3 - 2)$$

$$= 1(1) - 1(1) + 1(1)$$

$$= 1 - 1 + 1$$

$$|A| = 1 \quad |A| \neq 0$$

It is a linearly independent.

② determine whether the vectors $u = (1, 1, 2)$, $v = (1, 0, 1)$ & $w = (2, 1, 3)$ span the vector space $\mathbb{R}^3$.

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 1(0 - 1) - 1(3 - 2) + 2(1 - 0) \\ &= 1(-1) - 1(1) + 2(1) \end{aligned}$$

$$= -1 - 1 + 2$$

$$|A| = 0$$

It is linearly dependent.

④. The vectors $u_1 = (2, -3, 1)$, $u_2 = (1, 4, -2)$, $u_3 = (-8, 12, 7)$

$u_4 = (1, 3, 7, -17)$ & $u_5 = (-3, -5, 8)$ generate $\mathbb{R}^3$

Find a subset of the set $\{u_1, u_2, u_3, u_4, u_5\}$ that

Sol select the 3 vectors $\{u_1, u_2, u_3\}$

$$u_1 = (2, -3, 1)$$

$$u_2 = (1, 4, -2)$$

$$u_3 = (-8, 12, 7)$$

$$|A| = \begin{vmatrix} 2 & 1 & -8 \\ -3 & 4 & 12 \\ 1 & -2 & 7 \end{vmatrix}$$

$$= 2(28 - 10) - 1(21 - 7) - 8(21 - 4)$$

$$= 2(18) - 1(14) - 8(17)$$

$$= 36 - 14 - 136$$

$$= 2(32 - 10) - 1(-24 + 7) - 3(6 - 4)$$

$$= 2(22) - 1(-17) - 3(2)$$

$$= 44 + 17 - 6$$

$$= 61 - 6$$

$$|A| = 55$$

$$|A| \neq 0$$

It is a linearly independent

It is a basis

⑤. Show that the vectors ~~following~~ following set are bases for $\mathbb{R}^3$ or not. $\{(1, -3, -2), (-3, 1, 5), (-2, -10, -2)\}$.

Sol

$$|A| = \begin{vmatrix} 1 & -3 & -2 \\ -3 & 1 & 5 \\ -2 & -10 & -2 \end{vmatrix}$$

$$= 1(-2 + 30) + 3(6 + 10) - 2(30 + 2)$$

$$= 1(28) + 3(16) - 2(32)$$

$$= 28 + 48 - 64 = 72 - 64 = 8 \neq 0 \text{ It is dependent.}$$

Linear Dependence and Independence

Let V be a vector space over a field F . A finite number of distinct vectors $v_1, v_2, v_3, \dots, v_n$ in V is said to be linearly independent if

$$d_1 v_1 + d_2 v_2 + d_3 v_3 + \dots + d_n v_n = 0$$

$$d_1 = d_2 = d_3 = \dots = d_n = 0.$$

If $v_1, v_2, \dots, v_n$ are not linearly independent, ^{$|A| \neq 0$} then they are said to be linearly dependent.

$v_1, v_2, \dots, v_n$ are linearly dependent if ^{$|A| = 0$} if

$$d_1 v_1 + d_2 v_2 + \dots + d_n v_n = 0 \Rightarrow d_1, d_2, \dots, d_n \text{ not all zero.}$$

Q. Prove that the vectors $(1, 2, 1)$, $(2, 1, 0)$ and $(1, -1, 2)$ in $V_3(R)$ are linearly dependent.

sol

$$v_1 = (1, 2, 1)$$

$$v_2 = (2, 1, 0)$$

$$v_3 = (1, -1, 2)$$

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & -1 & 2 \end{vmatrix} = 1(2-0) - 2(4-0) + 1(-2-1)$$

$$= 2 - 8 - 3$$

$$= 2 - 11$$

$$|A| = -9$$

$$|A| \neq 0$$

It is a linearly independent

② determine whether the vector $v_1 = (1, -2, 3)$, $v_2 = (5, 6, -1)$ and $v_3 = (3, 2, 1)$ are linearly independent or linearly dependent set in $\mathbb{R}^3$.

sol Given

$$v_1 = (1, -2, 3)$$

$$v_2 = (5, 6, -1)$$

$$v_3 = (3, 2, 1)$$

Let

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & -2 & 3 \\ 5 & 6 & -1 \\ 3 & 2 & 1 \end{vmatrix} = 1(6+2) + 2(5+3) + 3(10-18) \\ &= 1(8) + 2(8) + 3(-8) \\ &= 8 + 16 - 24 \\ &= 24 - 24 \\ &= 0 \end{aligned}$$

$$|A| = 0$$

It is a linearly dependent.

③ why $v_1 = (-1, 2, 4)$, $v_2 = (5, -10, -20)$ in $\mathbb{R}^3$ are linearly independent in $\mathbb{R}^3$.

sol

Given

$$v_1 = (-1, 2, 4)$$

$$v_2 = (5, -10, -20)$$

$$v_3 = (0, 0, 0)$$

$$\begin{aligned} |A| &= \begin{vmatrix} -1 & 2 & 4 \\ 5 & -10 & -20 \\ 0 & 0 & 0 \end{vmatrix} = -1(0-0) - 2(0-0) + 4(0-0) \\ &= 0 \end{aligned}$$

$$|A| = 0$$

It is a linearly dependent

$\mathbb{R}^3$ are linearly independent in $\mathbb{R}^3$

④ Determine whether $(1, -1, 2)$, $(1, 2, 1)$ and $(1, 1, 4)$ are linearly independent or linearly dependent.

sol

sol

$$v_1 = (1, -1, 2)$$

$$v_2 = (1, 2, 1)$$

$$v_3 = (1, 1, 4)$$

$$|A| = \begin{vmatrix} 1 & -1 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 4 \end{vmatrix}$$

$$= 1[8-1] + 1[4-2] + 2[1-2]$$

$$= 1[7] + 1[2] + 2[-1]$$

$$= 7 + 2 - 2$$

$$= 7$$

$$= 7$$

It is a linearly independent.

⑤ Verify whether the vector $(2, 7)$ and $(12, 42)$ are linearly independent or linearly dependent.

sol

$$x_1 = (2, 7)$$

$$x_2 = (12, 42)$$

$$|A| = \begin{vmatrix} 2 & 12 \\ 7 & 42 \end{vmatrix} = 84 - 84 = 0$$

$$|A| = 0$$

H.W It is a linearly dependent.

⑥ Determine whether the vectors $(1, -1, 2)$, $(1, -2, 1)$ and $(1, 1, 4)$ in $\mathbb{R}^3$ are linearly independent or linearly dependent.

⑦ verify whether the vector $(1, 2, 3)$, $(3, -2, 1)$ and $(1, -6, -5)$ are linearly independent or not.

⑧ Are the vector $v_1 = (2, 1, 14)$, $v_2 = (7, -3, 15)$ and $v_3 = (-1, 1, 7)$ are linearly independent.

⑨. Verify whether the vector $(1, 1, 0, 0)$, $(0, 1, -1, 0)$ and $(0, 0, 0, 3)$ are linearly independent or linearly dependent.

Sol

$$v_1 = (1, 1, 0, 0)$$

$$v_2 = (0, 1, -1, 0)$$

$$v_3 = (0, 0, 0, 3)$$

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$$

$$a_1(1, 1, 0, 0) + a_2(0, 1, -1, 0) + a_3(0, 0, 0, 3) = (0, 0, 0, 0)$$

$$a_1 + 0 + 0 = 0 \Rightarrow \boxed{a_1 = 0}$$

$$a_1 + a_2 + 0 = 0 \Rightarrow a_1 + a_2 = 0 \quad 0 + a_2 = 0 \Rightarrow \boxed{a_2 = 0}$$

$$0 + a_2 + 0 = 0$$

$$\boxed{a_2 = 0}$$

$$0 + 0 + 3a_3 = 0$$

$$3a_3 = 0$$

$$\boxed{a_3 = 0}$$

$$a_1 = a_2 = a_3 = 0$$

It is a linearly dependent

v_1, v_2, v_3 are linearly independent.

⑩^x Verify whether the vectors $v_1 = (1, 1, 2, 4)$, $v_2 = (2, -1, -5, 2)$, $v_3 = (1, -1, -4, 0)$ and $v_4 = (2, 1, 1, 6)$ are linearly independent or linearly dependent.

sol

$$v_1 = (1, 1, 2, 4)$$

$$v_2 = (2, -1, -5, 2)$$

$$v_3 = (1, -1, -4, 0)$$

$$v_4 = (2, 1, 1, 6)$$

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{bmatrix}$$

$$= 1 \begin{vmatrix} -1 & -1 & 1 \\ -5 & -4 & 1 \\ 2 & 0 & 6 \end{vmatrix} - 2 \begin{vmatrix} 1 & -1 & 1 \\ 2 & -4 & 1 \\ 4 & 0 & 6 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 & 1 \\ 2 & -5 & 1 \\ 4 & 2 & 6 \end{vmatrix}$$

$$- 2 \begin{vmatrix} 1 & -1 & -1 \\ 2 & -5 & -4 \\ 4 & 2 & 0 \end{vmatrix}$$

$$= +1 (-1(-24-0) + 1(-30-2) + 1(0+8))$$

$$- 2 (1(-24-0) + 1(12-4) + 1(0+16))$$

$$+ 1 [1(-30-2) + 1(12-4) + 1(4+20)]$$

$$- 2 [1(0+8) + 1(0+16) - 1(4+20)]$$

$$= 1(-1(-24) + 1(-32) + 1(8))$$

$$- 2(-24 + 8 + 16) + 1(-32 + 8 + 24) - 2(8 + 16 - 24)$$

$$= 24 - 32 + 8 + 24 - 16 - 32 - 32 + 8 + 24$$

$$= 48 + 48 + 48 + 16 - 16 - 16 - 32 - 32 - 32 - 32$$

$$= 144 - 16 - 128$$

$$= 144 - 144 = 0 \quad |A| = 0$$

It is a linearly dependent.

- ⑩. Find the values of k if the vectors $v_1 = (k, -1, 3)$
 $v_2 = (3, 4, -1)$ and $v_3 = (3, 2, 1)$ are linearly independent.

sol

$$|A| = \begin{vmatrix} k & 3 & 3 \\ -1 & 4 & 2 \\ 3 & -1 & 1 \end{vmatrix}$$

$$= k(4-2) - 3(-1-6) + 3(1-12) = 0$$

$$= k(2) - 3(-7) + 3(-11) = 0$$

$$2k + 21 - 33 = 0$$

~~$$2k + 21 - 33 = 0$$~~

~~$$2k = 12$$~~

$$6k + 21 - 33 = 0$$

$$6k = 33 - 21$$

$$6k = 12$$

$$k = 2$$

$$k = 2$$

$$\begin{array}{r} 33 \\ 21 \\ \hline 12 \end{array}$$

It is a linearly independent.

H.M.

- ⑫. If the vectors $(0, 1, a)$, $(1, a, 1)$ and $(a, 1, 0)$ of $V_3(K)$ are linearly dependent, find the value of 'a'.

Ans: $a = 0, a = \pm\sqrt{2}$

- ⑬. If $V = P^4$ is a vector space over the field F ; then verify whether the vector $(1, 0, 1, 0)$, $(0, 1, 0, 1)$ and $(0, 0, 0, 2)$ are linearly independent or linearly dependent.

14. determine whether the following set
 $\{x^3+2x^2, -x^2+3x+1, x^3-x^2+2x+1\}$ and $P_3(\mathbb{R})$ are
 linearly dependent or independent.

sol.

$$v_1 = x^3+2x^2 \Rightarrow (1, 2, 0, 0)$$

$$v_2 = -x^2+3x+1 \Rightarrow (0, -1, 3, 1)$$

$$v_3 = x^3-x^2+2x+1 \Rightarrow (1, -1, 2, 1)$$

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = 0.$$

$$a_1(1, 2, 0, 0) + a_2(0, -1, 3, 1) + a_3(1, -1, 2, 1) = (0, 0, 0, 0)$$

$$\begin{array}{l|l} a_1 + 0 + a_3 = 0 \rightarrow ① & 2a_1 - a_2 - a_3 = 0 \rightarrow ② \\ a_1 + a_3 = 0 & -2a_3 - a_2 - a_3 = 0 \\ & a_1 = -a_3 \\ & -a_2 - 3a_3 = 0 \end{array}$$

$$-a_2 = 3a_3$$

$$a_2 = -3a_3$$

$$0 + 3a_2 + 2a_3 = 0 \rightarrow ③$$

$$0 + a_2 + a_3 = 0 \rightarrow ④$$

$$\boxed{a_2 = a_3} \rightarrow ⑤$$

use ⑤ sub in ③

$$3a_3 + 2a_3 = 0$$

$$5a_3 = 0$$

$$\boxed{a_3 = 0} \rightarrow ⑥$$

use ⑥ sub in ①

$$a_1 + a_3 = 0$$

$$a_1 + 0 = 0$$

$$\boxed{a_1 = 0} \rightarrow ⑦$$

use ⑦ sub in ⑤

$$\boxed{a_2 = 0} \rightarrow ⑧$$

If P_3 is a linearly dependent / v_1, v_2, v_3 are linearly independent

⑤ If $V = P(x)$ is a vector space over a field F and $S = \{1, 1+x, 1+x+x^2\}$, prove that S is linearly independent.

sol

$$v_1 = 1 \Rightarrow (1, 0, 0)$$

$$v_2 = 1+x \Rightarrow (1, 1, 0)$$

$$v_3 = 1+x+x^2 \Rightarrow (1, 1, 1)$$

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 1(1-0) - 0(0) + 0(0)$$

$$|A| = 1$$

$$|A| \neq 0$$

It is a linearly independent.

Basis And Dimensions.

A basis of the vector space V is a linearly independent subset S which spans (generates) the whole space V .

①. prove that $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a basis for $V_3(R)$.

sol

$$\text{Given } S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$|A| = 1(1-0) - 0(0) + 0(0)$$

$$|A| = 1$$

$$|A| \neq 0$$

3 Ps are linearly independent
 $\therefore$ 3 Ps a basis for $V_3(P)$.

② check whether the vectors $v_1 = (1, -3, 2)$, $v_2 = (4, 1, 0)$
and $v_3 = (0, 2, -1)$ form a basis for P^3 .

sol

$$v_1 = (1, -3, 2)$$

$$v_2 = (4, 1, 0)$$

$$v_3 = (0, 2, -1)$$

$$|A| = \begin{vmatrix} 1 & 4 & 0 \\ -3 & 1 & 2 \\ 2 & 0 & -1 \end{vmatrix}$$

$$= 1[-1-0] - 4[-3-1] + 0(0)$$

$$= -1 - 4(-1)$$

$$= -1 + 4$$

$$= 3$$

$$|A| \neq 0$$

3 Ps are linearly independent.

3 Ps a basis for $V = P^3$.

①. Determine the basis and dimension of the solution space of the linear homogeneous equations $x+y-z=0$, $-2x-y+2z=0$ and $-x+z=0$.

Sol

$$x+y-z=0$$

$$-2x-y+2z=0$$

$$-x+z=0$$

$$AX=B$$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & +1 & -1 & 0 \\ -2 & -1 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

$$A \sim \left[\begin{array}{ccc|c} 1 & +1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{array} \right] \begin{array}{l} R_2 = R_2 + 2R_1 \\ R_3 = R_3 + R_1 \end{array}$$

$$A \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] R_3 = R_3 - R_2$$

$$x+y-z=0 \quad \text{--- (1)}$$

$$y=0 \quad \text{--- (2)}$$

$$y=0 \text{ sub in (1)}$$

$$x-z=0$$

$$\text{Put } x=k, z=k, y=0$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The set $S = \{1, 0, 1\}$ is a basis
Dimension = 1

② find the basis and dimension, of the solution space of homogeneous system $2x_1 + 2x_2 - x_3 + x_5 = 0$,
 $-x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0$, $x_1 + x_2 - 2x_3 - x_5 = 0$,
 $x_3 + x_4 + x_5 = 0$.

So Given

$$2x_1 + 2x_2 - x_3 + x_5 = 0$$

$$-x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0$$

$$x_1 + x_2 - 2x_3 - x_5 = 0$$

$$x_3 + x_4 + x_5 = 0$$

$$Ax = B$$

$$A = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 1 & -2 & 0 & -1 \\ -1 & -1 & 2 & -3 & 1 \\ 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \quad R_2 \sim R_3$$

$$2 \begin{bmatrix} 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 3 & 0 & 3 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{array}{l} R_2 = R_2 + R_1 \\ R_3 = R_3 - 2R_1 \end{array}$$

$$2 \begin{bmatrix} 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} R_3 = R_3 \cdot \frac{1}{3}$$

$$2 \begin{bmatrix} 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 4 & 0 \end{bmatrix} R_4 = R_4 - R_3$$

$$x_1 + x_2 - 2x_3 - x_5 = 0$$

$$-3x_4 = 0$$

$$x_3 + x_5 = 0$$

$$x_4 = 0$$

$$x_4 = 0$$

$$x_3 + x_5 = 0$$

$$\boxed{x_3 = -x_5}$$

$$x_1 + x_2 + 2x_5 - x_5 = 0$$

$$x_1 + x_2 + x_5 = 0$$

$$\boxed{x_1 = -x_2 - x_5}$$

$$x_1 = -x_2 - x_5$$

$$x_2 = x_2$$

$$x_3 = -x_5$$

$$x_4 = 0$$

$$x_5 = x_5$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_2 - x_5 \\ x_2 \\ -x_5 \\ 0 \\ x_5 \end{bmatrix} \Rightarrow \begin{bmatrix} -x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_5 \\ 0 \\ -x_5 \\ 0 \\ x_5 \end{bmatrix}$$

$$\Rightarrow x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

The set $S = \{(-1, 1, 0, 0, 0), (-1, 0, -1, 0, 1)\}$ is a basis of the homogeneous system

Dimension = 2

④ find a basis and dimension of the solution space of homogeneous system $x_1 + 3x_2 - 2x_3 + 2x_5 = 0$, $2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 0$, $5x_3 + 10x_4 + 15x_6 = 0$ and $2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 0$.

Sol

$$x_1 + 3x_2 - 2x_3 + 2x_5 = 0$$

$$2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 0$$

$$5x_3 + 10x_4 + 15x_6 = 0$$

$$2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 0$$

$$AX=B$$

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 1 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$$

$$A = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 1 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{array} \right]$$

$$A \sim \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 1 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 0 & 0 & 5 & 10 & 0 & 21 & 0 \end{array} \right] \quad R_4 = R_4 - R_3$$

$$\sim \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 0 & 0 & 5 & 10 & 0 & 21 & 0 \end{array} \right] \quad \begin{array}{l} R_2 = R_2 - 2R_1 \\ R_3 = \frac{R_3}{5} \end{array}$$

$$\sim \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 \end{array} \right] \quad R_4 = R_4 - 6R_3$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6 & 0 \end{bmatrix} R_3 = \frac{R_2}{5}$$

$$x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \rightarrow (1)$$

$$-x_3 - 2x_4 - 3x_6 = 0 \rightarrow (2)$$

$$x_6 = 0 \rightarrow (3)$$

eqn (2) Sub Pn (3)

$$x_6 = 0$$

$$-x_3 - 2x_4 + 0 = 0$$

$$-x_3 - 2x_4 = 0$$

$$x_3 + 2x_4 = 0$$

$$x_3 = -2x_4 \rightarrow (4)$$

$x_3 = -2x_4$ Sub Pn (1)

$$x_1 + 3x_2 - 2(-2x_4) + 2x_5 = 0$$

$$x_1 + 3x_2 + 4x_4 + 2x_5 = 0$$

$$x_1 = -3x_2 - 4x_4 - 2x_5 \rightarrow (5)$$

$$x_2 = x_2$$

$$x_3 = -2x_4$$

$$x_4 = x_4$$

$$x_5 = x_5$$

$$x_6 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3x_2 - 4x_4 - 2x_5 \\ x_2 \\ -2x_4 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -4x_4 \\ 0 \\ -2x_4 \\ x_4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2x_5 \\ 0 \\ 0 \\ 0 \\ x_5 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x^2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x^4 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x^5 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ one non-zero equation
5 unknowns

The set $S = \{(-3, 1, 0, 0, 0, 0), (-1, 0, -2, 1, 0, 0), (2, 0, 0, 0, 1, 0)\}$ is a basis of the homogeneous system.

Dimension = 3.

(6). Find the dimension and a basis for the solution of homogeneous equation $x_1 + 2x_2 + 2x_3 - x_4 + 3x_5 = 0$,

$x_1 + 2x_2 + 3x_3 + x_4 + x_5 = 0$, and $3x_1 + 6x_2 + 8x_3 + x_4 + 5x_5 = 0$

sol

$$x_1 + 2x_2 + 2x_3 - x_4 + 3x_5 = 0$$

$$x_1 + 2x_3 + 3x_4 + x_5 = 0$$

$$3x_1 + 6x_2 + 8x_3 + x_4 + 5x_5 = 0$$

$$AX = B$$

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 1 & 2 & 3 & 1 & 1 \\ 3 & 6 & 8 & 1 & 5 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = B$$

$$[A|B] = \begin{bmatrix} 1 & 2 & 2 & -1 & 3 & 0 \\ 1 & 2 & 3 & 1 & 1 & 0 \\ 3 & 6 & 8 & 1 & 5 & 0 \end{bmatrix}$$

$$\text{Ans} \begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 2 & 4 & -4 \end{bmatrix} \begin{array}{l} R_2 = R_2 - R_1 \\ R_3 = R_3 - 3R_1 \end{array}$$

$$\text{Ans} \begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ R_3 = \frac{R_3}{2} \end{array}$$

$$\text{Ans} \begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ \\ R_3 = R_3 - R_2 \end{array}$$

$$x_1 + 2x_2 + 2x_3 - x_4 + 3x_5 = 0$$

$$x_3 + 2x_4 - 2x_5 = 0$$

$$x_3 + 2x_4 - 2x_5 = 0$$

$$x_3 = -2x_4 + 2x_5$$

$$x_1 + 2x_2 + 2[-2x_4 + 2x_5] - x_4 + 3x_5 = 0$$

$$x_1 + 2x_2 - 4x_4 + 4x_5 - x_4 + 3x_5 = 0$$

$$x_1 + 2x_2 - 5x_4 + 7x_5 = 0$$

$$x_1 = -2x_2 + 5x_4 - 7x_5$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 + 5x_4 - 7x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 5x_4 \\ 0 \\ -2x_4 \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} -7x_5 \\ 0 \\ 2x_5 \\ 0 \\ x_5 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 5 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

The set $S = \{(-2, 1, 0, 0, 0), (5, 0, -2, 1, 0), (-7, 0, 2, 0, 1)\}$ is a basis
 Dimension = 3

⑥ Find the dimension and the basis for the solution space W of the system of homogeneous equation given below: $2x_1 + 2x_2 - x_3 + x_5 = 0$,
 $-x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0$, $x_1 + x_2 - 2x_3 - 5x_5 = 0$,
 $x_3 + x_4 + x_5 = 0$,

Sol

$$2x_1 + 2x_2 - x_3 + x_5 = 0$$

$$-x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0$$

$$x_1 + x_2 - 2x_3 - 5x_5 = 0$$

$$x_3 + x_4 + x_5 = 0$$

$$AX = B$$

$$A = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -5 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -5 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$A \sim \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -5 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array} \right] \begin{array}{l} R_2 \\ R_1 \\ R_3 \end{array}$$

$$A \sim \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -5 & 0 \\ 0 & 0 & 0 & -3 & -4 & 0 \\ 0 & 0 & 3 & 0 & 11 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array} \right] \begin{array}{l} R_1 = R_2 + R_1 \\ R_3 = R_3 - 2R_1 \end{array}$$

$$A \sim \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -5 & 0 \\ 0 & 0 & 0 & -3 & -4 & 0 \\ 0 & 0 & -1 & 0 & \frac{11}{3} & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array} \right] R_3 = \frac{R_3}{2}$$

$$A \sim \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -5 & 0 \\ 0 & 0 & 0 & -3 & -4 & 0 \\ 0 & 0 & 1 & 0 & \frac{11}{3} & 0 \\ 0 & 0 & 0 & 1 & -\frac{8}{3} & 0 \end{array} \right] R_4 = R_4 - R_3$$

$$1 - \frac{11}{3}$$

$$\frac{-8}{3}$$

$$x_1 + x_2 - 2x_3 - 5x_5 = 0 \rightarrow (1)$$

$$-3x_4 - 4x_5 = 0 \rightarrow (2)$$

$$x_3 + \frac{11}{3}x_5 = 0 \rightarrow (3)$$

$$x_4 - \frac{8}{3}x_5 = 0 \rightarrow (4)$$

from (2) $-3x_4 - 4x_5 = 0$

$$+3x_4 + 4x_5 = 0 \rightarrow (5)$$

from (3) $x_3 - \frac{8}{3}x_5 = 0$

$$3x_3 - 8x_5 = 0 \rightarrow (6)$$

$$3x_2 + 4x_5 = 0$$

$$\Rightarrow 3x_2 \stackrel{(-)}{-} 8x_5 = 0$$

$$12x_5 = 0$$

$$\boxed{x_5 = 0}$$

$x_5 = 0$ Sub Pn ③ & ④

$$\boxed{x_3 = 0}$$

$$\boxed{x_4 = 0}$$

$$x_1 + x_2 - 2x_3 - 5x_4 = 0$$

$$x_1 + x_2 = 0$$

$$\boxed{x_1 = -x_2}$$

$$x_1 = -x_2$$

$$x_2 = x_2$$

$$x_3 = 0$$

$$x_4 = 0$$

$$x_5 = 0$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The set $S = \{(-1, 1, 0, 0, 0)\}$ is a basis.

Dimension = 1

①. Is it possible for a vector u to have two different negative vectors? Justify.

Sol.

Given U is a vector space V .

It is not possible for a vector u to have two different negative vectors.

Let u' & u'' are two negative vectors of u .

Then $u + u' = 0 \rightarrow \text{①}$

$u + u'' = 0 \rightarrow \text{②}$

① - ②

$$u + u' = u + u''$$

$$u' = u + u'' - u$$

$$\boxed{u' = u''}$$

$\therefore$ Negative vector is unique.

$\therefore$ It is not possible for a vector u to have two different negative vectors.

②. Does a line passing through origin of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$?

(or)

Show that a line through the origin is a subspace of $\mathbb{R}^3$.

Sol.

A line passing through the origin in $\mathbb{R}^3$ is

$$\frac{x-0}{a} = \frac{y-0}{b} = \frac{z-0}{c} \quad \boxed{a, b, c \neq 0}$$

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = k \text{ (say)}$$

$$(x, y, z) = (ak, bk, ck)$$

$$\therefore W = \{(ak, bk, ck) \mid k \in \mathbb{R}\}$$

$\therefore W$ is a subspace of $\mathbb{R}^3$

analogously,

w is a non-empty

$$u = (a_1, b_1, c_1)$$

$$v = (a_2, b_2, c_2)$$

$$au + bv = a(a_1, b_1, c_1) + b(a_2, b_2, c_2)$$

$$= a(a_1 + ba_2, b_1 + bb_2, c_1 + bc_2) \in w$$

$$au + bv \in w$$

$\therefore w$ is a non-empty

$\therefore w$ is a subspace of $\mathbb{R}^3$

ie) a line passing through the origin of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

3. State Dimension Theorem in linear algebra (or) Rank-Nullity theorem.

Statement

Let V & W be vector spaces &

$T: V \rightarrow W$ be linear transformation.

If V is finite dimensional, then

$$\text{Nullity}(T) + \text{rank}(T) = \dim V.$$

4. Define Dimension (or) Finite dimension.

Let V be a vector space over a field F , the number of elements in any basis of $V(F)$.

It is denoted by $\dim V$

Eg:

$$1] \dim \{0\} = 0$$

$$2] \dim F^n = n$$

$$3] \dim M_{m \times n}(F) = mn$$

$$4] \dim P_n(F) = n+1$$

⑤. What are the possible subspaces of $\mathbb{R}^2$.

Sol

- i. $\{0\}$ is the subspace of $\mathbb{R}^2$
- ii. $\mathbb{R}^2$ is the subspace of $\mathbb{R}^2$
- iii. Lines through the origin are subspace of $\mathbb{R}^2$.

⑥. Prove that the union of two subspaces of a vector space is subspace.

Sol

$$\text{Let } A = \{ (a, 0, 0) \mid a \in \mathbb{R} \}$$

$$B = \{ (0, b, 0) \mid b \in \mathbb{R} \}$$

$$\begin{aligned} A+B &= (a, 0, 0) + (0, b, 0) \\ &= (a, b, 0) \end{aligned}$$

$$A+B = W$$

W is a subspace of $\mathbb{R}^3$.

⑦ Let $V = \{ (a_1, a_2) \mid a_1, a_2 \in F \}$ where F is a field. Define addition of element of V & for $c \in F$ and $(a_1, a_2) \in V$, define $c(a_1, a_2) = (ca_1, ca_2)$. Is V a vector space over F with those operations.

Sol

$$\text{Given } V = \{ (a_1, a_2) \mid a_1, a_2 \in F \}$$
$$c(a_1, a_2) = (ca_1, ca_2) \text{ for } c \in F, (a_1, a_2) \in V$$

$$\therefore 0(a_1, a_2) = (0a_1, 0a_2) \text{ is zero vector.}$$

$\Rightarrow$ which contradiction rule 1.

The vector 0 is unique

$\therefore V$ is not a vector space

$$\exists \text{ an element } 0 \ni : x+0 = x, \forall x \in V$$

7] If $V = A \oplus B$, then show that $\dim V = \dim A + \dim B$

sol Given $V = A \oplus B$

$$A \cap B = \{0\} \quad A \cup B = \{0\}$$

$$\dim(A \cap B) = 0$$

we know that $A \cap B = A + B - (A \cup B)$

$$\dim(A \cap B) = \dim A + \dim B - \dim(A \cup B)$$

$$0 = \dim A + \dim B - \dim V$$

by $A \cup B = V$

$$\dim V = \dim A + \dim B$$

Hence the proof.

Q. Define vector space or linear space.

A vector space V over a field F consists of a set on which two operations $(+ \text{ \& } \cdot)$ are defined so that for each pair of element $(x, y) \in V$, there is a unique element $x+y$ in V and for each element $a \in F$ & each element $x \in V$ there is a unique element ax in V .

Axioms (or) properties:

① commutative law

$$\forall x, y \in V, \quad x+y = y+x$$

② Associativity law.

$$1) \quad \forall x, y, z \in V, \quad \exists; (x+y)+z = x+(y+z)$$

$$2) \quad \exists 0 \in V \text{ s.t. } x+0 = x, \quad \forall x \in V$$

$$3) \quad \forall x \in V, \quad \exists y \in V \text{ s.t. } x+y = 0$$

$$4) \quad \forall x \in V, \quad x = x$$

$$5) \quad \forall a, b \in F \text{ \& } x \in V; (ab)x = a(bx)$$

$$6) \quad \forall a \in F \text{ \& } \forall x, y \in V; a(x+y) = ax + ay$$

$$7) \quad \forall a, b \in F \text{ \& } \forall x \in V; (a+b)x = ax + bx.$$

Q. verify cauchy schwarz inequality for $u = (1, -1, 3)$ and $v = (2, 0, -1)$

Sol $u = (1, -1, 3)$ & $v = (2, 0, -1)$

By cauchy schwarz inequality.

$$|\langle u, v \rangle| \leq \|u\| \|v\|$$

$$\langle u, v \rangle = (x_1 x_2, y_1 y_2, z_1 z_2)$$

$$\begin{aligned} \langle u, v \rangle &= (1, -1, 3) (2, 0, -1) \\ &= 2 - 0 - 3 \end{aligned}$$

$$\langle u, v \rangle = -1$$

$$|\langle u, v \rangle| = |-1| = 1$$

$\Rightarrow \text{Q.E.D.}$

$$\|u\|^2 = (u, u)$$

$$= (1, -1, 3) (1, -1, 3)$$

$$= 1 + 1 + 9$$

$$\|u\|^2 = 11 \Rightarrow \|u\| = \sqrt{11}$$

$$\|v\|^2 = (v, v) = (2, 0, -1) (2, 0, -1)$$

$$= 4 + 0 + 1$$

$$\|v\|^2 = 5$$

$$\|v\| = \sqrt{5}$$

$$\|u\| + \|v\| = \sqrt{u} + \sqrt{v} \rightarrow \textcircled{a}$$

① & ②

$$1 \leq \sqrt{u} + \sqrt{v}$$

$$|(u,v)| \leq \|u\| \|v\|.$$

7. Consider V be the vector of polynomials with an inner product $\langle f, g \rangle = \int_0^1 f(t)g(t) dt$. Let $f(t)$

$$f(t) = t+2, g(t) = t^2-2t-3. \text{ find i) } \langle f, g \rangle \text{ \& ii) } \|f\|$$

sol

$$V = \{f_n(x) \mid x \in \mathbb{R}\} \rightarrow \textcircled{a}$$

$$\langle f, g \rangle = \int_0^1 f(t)g(t) dt$$

$$f(t) = t+2 \rightarrow \textcircled{b}$$

$$g(t) = t^2-2t-3 \rightarrow \textcircled{c}$$

i] To find $\langle f, g \rangle$

$$\langle f, g \rangle = \int_0^1 f(t)g(t) dt$$

$$= \int_0^1 (t+2)(t^2-2t-3) dt$$

$$= \int_0^1 t^3 - 2t^2 - 3t + 2t^2 - 4t - 6 dt$$

$$= \left[\frac{t^4}{4} - \frac{7t^2}{2} - 6t \right]_0^1$$

$$= \left[\frac{1}{4} - \frac{7}{2} - 6 \right]$$

$$= \frac{2-28-24}{8}$$

$$= \frac{2-76}{8} = -\frac{37}{4}$$

$$\langle f, g \rangle = -\frac{37}{4}$$

if to find $\|g\|$

By the definition $\|g\| = \sqrt{(f, f)}$.

$$\|g\|^2 = (f, f)$$

$$= \int_0^1 (f(t) - f(1))^2 dt = \int_0^1 (t+2)^2 dt$$

$$= \int_0^1 t^2 + 4 + 2 \cdot 2t dt$$

$$= \left[\frac{t^3}{3} + 4t + 2 \cdot \frac{t^2}{2} \right]_0^1$$

$$= \left[\frac{1}{3} + 4 + 2 \right]$$

$$= \frac{1+12+6}{3}$$

$$\|g\|^2 = \frac{19}{3}$$

(A)

⑥ Prove that $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ for all $x, y \in V$ (Parallelogram law).

$$\|x\|^2 = (x, x)$$

$$\|x+y\|^2 = (x+y, x+y)$$

$$= (x, x+y) + (y, x+y)$$

$$= (x, x) + (x, y) + (y, x) + (y, y)$$

$$\|x+y\|^2 = \|x\|^2 + (x, y) + (y, x) + \|y\|^2 \rightarrow \textcircled{1}$$

$$\|x-y\|^2 = (x-y, x-y)$$

$$= (x, x-y) - (y, x-y)$$

$$= (x, x) - (x, y) - (y, x) + (y, y)$$

$$\|x-y\|^2 = \|x\|^2 - (x, y) - (y, x) + \|y\|^2 \rightarrow \textcircled{2}$$

$$\|x+y\|^2 + \|x-y\|^2 = \|x\|^2 + \cancel{(x, y)} + \cancel{(y, x)} + \|y\|^2 + \|x\|^2 - \cancel{(x, y)} - \cancel{(y, x)} + \|y\|^2$$

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

⑦ Let $f(t) = t+2$ and $g(t) = 3t-2$ be the polynomials in $P(t)$ and $\langle f, g \rangle = \int_0^1 f(t)g(t) dt$. Find i) $\langle f, g \rangle$ & ii) $\|f\|$. iii) $\|g\|$ iv) normalize f and g .

31

Given $f(t) = t+2$

$$g(t) = 3t-2$$

$$\langle f, g \rangle = \int_0^1 f(t)g(t) dt$$

ii] to find $\langle f, g \rangle$

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt = \int_0^1 (t+2)(3t-2)dt$$

$$= \int_0^1 (3t^2 - 2t + 6t - 4)dt$$

$$= \left[\frac{3t^3}{3} - \frac{2t^2}{2} + \frac{6t^2}{2} - 4t \right]_0^1$$

$$= 1 - 1 + 3 - 4$$

$$= -1$$

$$\langle f, g \rangle = -1$$

iii] $\|f\|$

$$\|f\|^2 = (f, f)$$

$$= \int_0^1 (f(t))^2 dt$$

$$= \int_0^1 (t+2)^2 dt = \int_0^1 (t^2 + 4 + 2 \cdot 2t) dt$$

$$= \left[\frac{t^3}{3} + 4t + 2t^2 \right]_0^1$$

$$= \left[\frac{1}{3} + 4 + 2 \right]$$

$$= \frac{1+12+6}{3}$$

$$\|f\|^2 = 19/3$$

$$\|f\| = \sqrt{19/3}$$

iv] $\|g\|$

$$\|g\|^2 = (g, g)$$

$$= \int_0^1 (g(t))^2 dt = \int_0^1 (3t-2)^2 dt$$

$$= \int_0^1 (9t^2 + 4 - 2 \cdot 6t) dt$$

$$= \left[\frac{9t^3}{3} + 4t - \frac{6t^2}{2} \right]_0^1$$

$$= [3 + 4 - 6]$$

$$\|g\|^2 = 1$$

$$\|g\| = \sqrt{1} \quad \|g\| = 1$$

iv) Normalization of f and g

Normalization of f is $\hat{f} = \frac{f}{\|f\|} = \frac{t+2}{\sqrt{19/3}}$

Normalization of g is $\hat{g} = \frac{g}{\|g\|} = \frac{3t-2}{1} = 3t-2$

Q. Consider the Inner Product Space V . Show that

$\|x+y\|^2 = \|x\|^2 + 2\operatorname{Re}\langle x, y \rangle + \|y\|^2$ for all $x, y \in V$. $\operatorname{Re}\langle x, y \rangle$ denotes the real part of the complex number $\langle x, y \rangle$.

Sol

$\|x\|^2 = \langle x, x \rangle$

$\|x+y\|^2 = \langle x+y, x+y \rangle$

$= \langle x, x+y \rangle + \langle y, x+y \rangle$

$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$

$= \langle x, x \rangle + \langle x, y \rangle + \overline{\langle x, y \rangle} + \langle y, y \rangle$

$= \|x\|^2 + 2\operatorname{Re}\langle x, y \rangle + \|y\|^2 \quad \because z + \bar{z} = 2\operatorname{Re}(z)$

Q. For the Inner Product Space V , show that

$\|x\| - \|y\| \leq \|x-y\|$ for all $x, y \in V$.

Sol

T.I

Using Sub

$\|x+y\| \leq \|x\| + \|y\|$

$\|x\| - \|y\| \leq \|(x-y) + y\| - \|y\|$

$\leq \|x-y\| + \|y\| - \|y\|$

$\leq \|x-y\|$

$\|x\| - \|y\| \leq \|x-y\|$

$|\|x\| - \|y\|| \leq \|x-y\|$

Q. For the Inner Product Space V over F , show that

$\langle x, y \rangle = \frac{1}{4} \|x+y\|^2 - \frac{1}{4} \|x-y\|^2, x, y \in V$ and $F = \mathbb{R}$.

Sol

$\|x+y\|^2 = \langle x+y, x+y \rangle$

$= \langle x, x+y \rangle + \langle y, x+y \rangle$

$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$

$$\|x-y\|^2 = \langle x-y, x-y \rangle$$

$$= \langle x, x-y \rangle - \langle y, x-y \rangle$$

$$\|x-y\|^2 = \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle$$

$$= \frac{1}{4} \|x+y\|^2 - \frac{1}{4} \|x-y\|^2$$

$$= \frac{1}{4} \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle - \frac{1}{4} \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle$$

$$= \frac{1}{4} [\langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle - \langle x, x \rangle + \langle x, y \rangle]$$

$$= \frac{1}{4} [\langle x, y \rangle + \langle y, x \rangle + \langle x, y \rangle + \langle y, x \rangle]$$

$$= \frac{1}{4} [2 \langle x, y \rangle + 2 \langle y, x \rangle]$$

$$= \frac{1}{2} [\langle x, y \rangle + \langle y, x \rangle]$$

$$= \frac{1}{2} [\langle x, y \rangle + \overline{\langle y, x \rangle}] \quad \overline{\langle y, x \rangle} = \langle x, y \rangle$$

$$= \frac{1}{2} [\langle x, y \rangle + \langle x, y \rangle] \quad F = R$$

$$= \frac{1}{2} \cdot 2 \langle x, y \rangle$$

$$= \langle x, y \rangle$$

$$\frac{1}{4} \|x+y\|^2 - \frac{1}{4} \|x-y\|^2 = \langle x, y \rangle$$

⑪ Let V be an Inner Product space. Let x and y be orthogonal vectors in V . Prove that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

Sol.

$$\|x+y\|^2 = \langle x+y, x+y \rangle$$

$$= \langle x, x+y \rangle + \langle y, x+y \rangle$$

$$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$$

Given x and y are orthogonal

$$\langle x, y \rangle = 0, \langle y, x \rangle = 0$$

$$\|x+y\|^2 = \langle x, x \rangle + \langle y, y \rangle \quad \|x+y\|^2 = \|x\|^2 + \|y\|^2$$

②. Find the norm of $u = (3, 4, 10)$ in $\mathbb{R}^3$ with standard inner product.

$$\|u\|^2 = \langle u, u \rangle = (3, 4, 10) \cdot (3, 4, 10)$$

$$= 9 + 16 + 100$$

$$= 125$$

$$\|u\|^2 = 125$$

$$\|u\| = 5$$

$$\begin{array}{r} 125 \\ 124 \\ \hline 1 \end{array}$$

③. Find the value $\|3d\|$ of the vector $d = (1, 2, 3, 4)$

Sol

$$d = (1, 2, 3, 4) \text{ in } \mathbb{R}^4$$

$$3d = (3, 6, 9, 12)$$

$$\|3d\|^2 = (3, 6, 9, 12) \cdot (3, 6, 9, 12)$$

$$= 9 + 36 + 81 + 144$$

$$\|3d\|^2 = 270$$

$$\|3d\| = \sqrt{270}$$

Inner Product spaces

① Define Inner product?

Let V be a vector space over a field F . An inner product on V is a function which assigns to each ordered pair of vectors u, v in V a scalar α in F , denoted by $\langle u, v \rangle$

$$i] \langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle, w \in V$$

$$ii] \langle \alpha u, v \rangle = \alpha \langle u, v \rangle, \alpha \in F$$

$$iii] \langle u, v \rangle = \overline{\langle v, u \rangle}, \overline{\langle v, u \rangle} \text{ is the conjugate of } \langle v, u \rangle$$

$$iv] \langle u, v \rangle \geq 0 \text{ and } \langle u, v \rangle = 0 \text{ iff } u = 0$$

② Inner product space.

A vector space V (over a field F) with an inner product defined on it is called an inner product space.

③ Define standard inner product.

Standard inner product of F^n is usually called the dot product (or scalar product) denoted by $x \cdot y$ instead of $\langle x, y \rangle$ is defined to be

$$x \cdot y = \langle x, y \rangle = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

$x = (x_1, x_2, \dots, x_n)$ & $y = (y_1, y_2, \dots, y_n)$ are in F^n

$$a_i \in F, b_i \in F, i = 1, 2, \dots, n$$

④. Frobenius inner product

Let $V = M_{n \times n}(F)$ define

$$\langle A, B \rangle = \text{tr}(B^t A), A, B \in V$$

B^t is the transpose of the matrix B

$$\langle A, A \rangle = \text{tr}(A^t A) = \sum_{i=1}^n (A^t A)_{ii}$$

$$= \sum_{i=1}^n \sum_{k=1}^n (A^t)_{ik} A_{ki}$$

$$= \sum_{i=1}^n \sum_{k=1}^n \overline{A_{ki}} A_{ki}$$

$$= \sum_{i=1}^n \sum_{k=1}^n |A_{ki}|^2$$

This inner product is called Frobenius inner product.

⑤ The conjugate transpose.

The conjugate transpose of A is $\overline{A}^t$ and

is denoted by A^* (i.e.) $A^* = \overline{A}^t$ for all i, j .

①. Let $V(\mathbb{R})$ be a vector space of polynomials with inner product defined by $\langle f, g \rangle = \int_0^1 f(t)g(t) dt$. If $f(x) = x^2 + x - 4$ and $g(x) = x - 1$, find $\langle f, g \rangle$.

Sol $f(x) = x^2 + x - 4$, $g(x) = x - 1$

$$\langle f, g \rangle = \int_0^1 f(t)g(t) dt$$

$$\langle f, g \rangle = \int_0^1 (t^2 + t - 4)(t - 1) dt$$

$$= \int_0^1 (t^3 - t^2 - t + 4) dt$$

$$= \int_0^1 (t^3 - 5t + 4) dt \Rightarrow \left[\frac{t^4}{4} - 5\frac{t^2}{2} + 4t \right]_0^1$$

$$= \left[\frac{1}{4} - \frac{5}{2} + 4 \right]$$

$$= \frac{2 - 20 + 32}{8} = \frac{14}{8}$$

$$\langle f, g \rangle = \frac{7}{4}$$

②. Let V be the vector space of polynomials with inner product given by $\langle f, g \rangle = \int_0^1 f(t)g(t) dt$, $f(t) = t + 2$, $g(t) = t^2 - 2t - 3$. Find $\langle f, g \rangle$.

Sol $f(t) = t + 2$, $g(t) = t^2 - 2t - 3$.

$$\langle f, g \rangle = \int_0^1 f(t)g(t) dt$$

$$= \int_0^1 (t + 2)(t^2 - 2t - 3) dt$$

$$= \int_0^1 (t^3 - 2t^2 - 3t + 2t^2 - 4t - 6) dt$$

$$= \int_0^1 (t^3 - 7t - 6) dt$$

$$= \left[\frac{t^4}{4} - \frac{7t^2}{2} - 6t \right]_0^1$$

$$= \left[\frac{1}{4} - \frac{7}{2} - 6 \right] = \frac{2 - 28 - 48}{8} = \frac{-74}{8} = -\frac{37}{4}$$

(3). Let P^2 have the weighted Euclidean inner product defined as $\langle u, v \rangle = 2u_1v_1 + 3u_2v_2$ where $u = (1, 1)$ and $v = (3, 2)$. Let $w = (0, -1)$, compute $\langle u, v, 3w \rangle$.

Sol

$$\begin{array}{l|l} \langle u, v \rangle = 2u_1v_1 + 3u_2v_2 & 3w = 3(0, -1) \\ u = (1, 1), v = (3, 2), w = (0, -1) & 3w = (0, -3) \\ u+v = (1, 1) + (3, 2) & \\ = 4+3 & \\ u+v = 4+3 & \end{array}$$

Now $\langle u+v, 3w \rangle = 2(u+v)_1(3w)_1 + 3(u+v)_2(3w)_2$ $\because \langle u, v \rangle = 2u_1v_1 + 3u_2v_2$

$$\begin{aligned} &= (4+3), (0, -3) \\ &= 2(4)(0) - 3(3)(3) \\ &= 0 - 27 \end{aligned}$$

$$\langle u+v, 3w \rangle = -27$$

Angle between the vectors And orthogonal vectors

Angle between the two vectors u and v is determined by

$$\cos \theta = \frac{\langle u, v \rangle}{\|u\| \|v\|}$$

The two vectors u and v are perpendicular (or) orthogonal if and only if $\langle u, v \rangle = 0$.

(4). Let P_2 have the inner product $\langle p, q \rangle = \int_{-1}^1 p(x)q(x)dx$. Find the angle between p and q , where $p=x$, $q=x^2$ with respect to inner product $\langle p, q \rangle$.

Sol

$$\langle p, q \rangle = \int_{-1}^1 p(x)q(x)dx$$

$$p=x, q=x^2$$

$$\cos \theta = \frac{\langle p, q \rangle}{\|p\| \|q\|}$$

$$\langle p, q \rangle = \int_{-\frac{1}{4}}^{\frac{1}{4}} p(x)q(x) dx = \int_{-\frac{1}{4}}^{\frac{1}{4}} x \cdot x^2 dx = \int_{-\frac{1}{4}}^{\frac{1}{4}} x^3 dx = \left[\frac{x^4}{4} \right]_{-\frac{1}{4}}^{\frac{1}{4}} = \frac{1}{4} - \frac{1}{4} = 0$$

$$\langle p, q \rangle = 0$$

$$\cos \theta = \frac{0}{\|p\| \|q\|}$$

$$\cos \theta = 0$$

$$\theta = 90^\circ$$

Here the vectors p and q are orthogonal since $\theta = 90^\circ$ so $\langle p, q \rangle = 0$.

③ Let $u = (2, 3, 5)$ and $v = (1, -4, 3)$ in $\mathbb{R}^3$. Find the angle between u and v .

Sol

$$u = (2, 3, 5) \text{ and } v = (1, -4, 3)$$

Angle between u & v is

$$\cos \theta = \frac{\langle u, v \rangle}{\|u\| \|v\|}$$

$$\langle u, v \rangle = (2, 3, 5) \cdot (1, -4, 3) = 2 + (-12) + 15$$

$$\langle u, v \rangle = 2 - 12 + 15 = 5$$

$$u = (2, 3, 5)$$

$$\|u\|^2 = (2, 3, 5) \cdot (2, 3, 5)$$

$$= 4 + 9 + 25$$

$$\|u\|^2 = 38$$

$$\|u\| = \sqrt{38}$$

$$v = (1, -4, 3)$$

$$\|v\|^2 = (1, -4, 3) \cdot (1, -4, 3)$$

$$= 1 + 16 + 9$$

$$\|v\|^2 = 26$$

$$\|v\| = \sqrt{26}$$

$$\cos \theta = \frac{\langle u, v \rangle}{\|u\| \|v\|}$$

$$\cos \theta = \frac{5}{\sqrt{38} \sqrt{26}}$$

③ Find the angle between the two vectors $u = (4, 3, 1, -2)$

and $v = (-2, 1, 2, 3)$

sol

$$u = (4, 3, 1, -2) \quad v = (-2, 1, 2, 3)$$

$$\cos \theta = \frac{\langle u, v \rangle}{\|u\| \|v\|}$$

$$\langle u, v \rangle = (4, 3, 1, -2) \cdot (-2, 1, 2, 3)$$

$$= -8 + 3 + 2 - 6 = -14 + 5 = -9$$

$$\langle u, v \rangle = -9$$

$$u = (4, 3, 1, -2)$$

$$\|u\|^2 = (4, 3, 1, -2) \cdot (4, 3, 1, -2)$$

$$= 16 + 9 + 1 + 4$$

$$\|u\|^2 = 30$$

$$\|u\| = \sqrt{30}$$

$$v = (-2, 1, 2, 3)$$

$$\|v\|^2 = \langle v, v \rangle$$

$$= (-2, 1, 2, 3) \cdot (-2, 1, 2, 3)$$

$$= 4 + 1 + 4 + 9$$

$$\|v\|^2 = 18$$

$$\|v\| = \sqrt{18}$$

$$\cos \theta = \frac{-9}{\sqrt{30} \sqrt{18}}$$

$$\cos \theta = \frac{-9}{\sqrt{30} \sqrt{18}}$$

④ Find k so that the vectors $u = (1, 2, k, 3)$ and $v = (3, k, 7, -5)$ are orthogonal.

sol

$$u = (1, 2, k, 3) \quad v = (3, k, 7, -5)$$

u & v are orthogonal.

$$\langle u, v \rangle = 0$$

$$(1, 2, k, 3) \cdot (3, k, 7, -5) = 0$$

$$3 + 2k + 7k - 15 = 0$$

$$3 + 9k - 15 = 0$$

$$9k = 15 - 3$$

$$9k = 12$$

$$k = \frac{12}{9}$$

$$k = \frac{4}{3}$$

⑤. Show that the vectors $u = (-2, 3, 1, 4)$ and $v = (1, 2, 0, -1)$ are orthogonal in $\mathbb{R}^4$

Sol

$$u = (-2, 3, 1, 4), v = (1, 2, 0, -1)$$

To prove u & v are orthogonal

$$\langle u, v \rangle = 0$$

$$\langle u, v \rangle = (-2, 3, 1, 4) \cdot (1, 2, 0, -1)$$

$$= -2 + 6 + 0 - 4$$

$$= -6 + 6$$

$$\langle u, v \rangle = 0$$

$\langle u, v \rangle = 0$, u and v are orthogonal.

⑥. Let $u = (1, 3, 5)$, $v = (4, 5, 5)$ in $\mathbb{R}^3$ find the inner product $\langle u, v \rangle$.

Sol

$$\text{Let } u = (1, 3, 5), v = (4, 5, 5)$$

$$\langle u, v \rangle = (1, 3, 5) \cdot (4, 5, 5)$$

$$= 4 + 15 + 25$$

$$\langle u, v \rangle = 44$$

u & v are orthogonal.

⑦ Evaluate $\langle 3u - 2v, w \rangle$ when $u = (1, 3, -1, 2)$,

$$v = (4, -2, 2, 1) \text{ \& } w = (5, -1, -2, 6)$$

Sol

$$u = (1, 3, -1, 2) \Rightarrow 3u = 3(1, 3, -1, 2)$$

$$3u = (3, 9, -3, 6)$$

$$v = (4, -2, 2, 1) \Rightarrow 2v = 2(4, -2, 2, 1)$$

$$2v = (8, -4, 4, 2)$$

$$3u - 2v = (3, 9, -3, 6) - (8, -4, 4, 2)$$

$$= (3-8, 9+4, -3-4, 6-2)$$

$$3u - 2v = (-5, 13, -7, 4)$$

$$\langle 3u - 2v, w \rangle = (-5, 13, -7, 4) \cdot (5, -1, -2, 6)$$

$$= -25 - 13 + 32 + 24$$

$$= -38 + 56$$

$$\langle 3u - 2v, w \rangle = 18$$

$$\begin{array}{r} 416 \\ 56 \\ 32 \\ \hline 18 \end{array}$$

⑧ Prove that the vectors $u = (1, 1)$ & $v = (1, -1)$ are orthogonal w.r.to the Euclidean inner product on $\mathbb{R}^2$.

Sol let $u = (1, 1)$, $v = (1, -1)$

$$u \cdot v = (1, 1) \cdot (1, -1) = 1 - 1 = 0$$

u & v are orthogonal $\langle u, v \rangle = 0$

⑨. verify that the set of vectors $\{(1, 0, -1), (2, 0, 2), (0, 5, 0)\}$ is orthogonal w.r.to the Euclidean inner product.

Sol. Prove that any two distinct vectors in S .

$$u \cdot v = (1, 0, -1) \cdot (2, 0, 2)$$

$$= 2 + 0 - 2$$

$$u \cdot v = 0$$

$$u \cdot w = (1, 0, -1) \cdot (0, 5, 0)$$

$$= 0 + 0 + 0$$

$$u \cdot v = 0$$

Hence the prove

Distance between two vectors

Formula:-

$$\|x - y\| = \sqrt{(x - y) \cdot (x - y)}$$

⑩ Given $x = (1, 2, 3)$ & $y = (1, 0, 1)$, find distance between x & y through inner product.

Sol let $x = (1, 2, 3)$ & $y = (1, 0, 1)$

$$x - y = (1, 2, 3) - (1, 0, 1)$$

$$x - y = (0, 2, 2)$$

$$\|x - y\| = \sqrt{(x - y) \cdot (x - y)}$$

$$= \sqrt{(0, 2, 2) \cdot (0, 2, 2)}$$

$$= \sqrt{0 + 4 + 4} = \|x - y\| = \sqrt{8}$$

②. Find the norm of $(3, 4, 0)$ in $\mathbb{R}^3$ with the standard inner product.

Sol

$$\text{let } u = (3, 4, 0)$$

$$\|u\|^2 = (u, u) = (3, 4, 0) \cdot (3, 4, 0)$$

$$= 9 + 16 + 0$$

$$\|u\|^2 = 25$$

$$\|u\| = 5$$

③. Let $A = \begin{bmatrix} 1 & 1+2i \\ 2 & 3+4i \end{bmatrix}$. Find the conjugate transpose.

Sol

$$\text{let } A = \begin{bmatrix} 1 & 1+2i \\ 2 & 3+4i \end{bmatrix}$$

$$\overline{A} = \begin{bmatrix} 1 & 1-2i \\ 2 & 3-4i \end{bmatrix}$$

conjugate transpose:

$$A^* = \begin{bmatrix} 1 & 2 \\ 1-2i & 3-4i \end{bmatrix}$$

Unit - VII
① ^{define} Inner Product space.

A vector space V (over a field F) with an inner product defined on it is called an inner product space.

② ^{define} Orthogonal.

Let V be an inner product space and let $x, y \in V$. x is said to be orthogonal to y if $\langle x, y \rangle = 0$

③. ^{define} orthogonal set

Let V be an inner product space. A set S of vectors in V is said to be an orthogonal set if any two distinct vectors in S are orthogonal.

④. ^{define} orthonormal set

A set S is said to be an orthonormal set if S is orthogonal and $\|x\| = 1$ for all $x \in S$

Gram Schmidt orthogonalization process

Gram Schmidt process:

$$v_1 = u_1$$

$$v_2 = u_2 - \frac{(u_2, v_1)}{\|v_1\|^2} v_1$$

$$v_3 = u_3 - \frac{(u_3, v_1)}{\|v_1\|^2} v_1 - \frac{(u_3, v_2)}{\|v_2\|^2} v_2$$

$$\|v_1\|^2 = (v_1, v_1)$$

$$\|v_2\|^2 = (v_2, v_2)$$

$$\|v_3\|^2 = (v_3, v_3)$$

orthogonal set = (v_1, v_2, v_3)

orthogonal basis $q_1 = \frac{v_1}{\|v_1\|}$

$$q_2 = \frac{v_2}{\|v_2\|}$$

$$q_3 = \frac{v_3}{\|v_3\|}$$

Q. Let $\mathbb{R}^3$ have the Euclidean inner product $\langle \cdot, \cdot \rangle$.

Gram-Schmidt process $\{u_1, u_2, u_3\}$ into an

orthogonal basis where $u_1 = (1, 1, 1)$; $u_2 = (0, 1, 1)$

$$u_3 = (0, 0, 1)$$

Sol

Given $u_1 = (1, 1, 1)$, $u_2 = (0, 1, 1)$, $u_3 = (0, 0, 1)$

$$v_1 = u_1 = (1, 1, 1)$$

$$\|v_1\|^2 = (v_1, v_1) = (1, 1, 1) \cdot (1, 1, 1)$$

$$= 1+1+1$$

$$= 3$$

$$\|v_1\|^2 = 3$$

$$\|v_1\| = \sqrt{3}$$

$$v_2 = u_2 - \frac{(u_2, v_1)}{\|v_1\|^2} v_1$$

$$(u_2, v_1) = (0, 1, 1) \cdot (1, 1, 1) \\ = 0 + 1 + 1$$

$$(u_2, v_1) = 2$$

$$= (0, 1, 1) - \frac{2}{2} (1, 1, 1)$$

$$(u_3, v_1) = (0, 0, 1) \cdot (1, 1, 1) \\ = 0 + 0 + 1$$

$$= (0, 1, 1) + (-\frac{2}{2}, -\frac{2}{2}, -\frac{2}{2})$$

$$(u_3, v_1) = 1$$

$$= (0 - \frac{2}{2}, 1 - \frac{2}{2}, 1 - \frac{2}{2})$$

$$(u_3, v_2) = (0, 0, 1) \cdot (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$$

$$= (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$$

$$= 0 + 0 + \frac{1}{3}$$

$$v_2 = (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$$

$$(u_3, v_2) = \frac{1}{3}$$

$$\|v_2\|^2 = (v_2, v_2) = (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}) \cdot (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$$

$$= \frac{4}{9} + \frac{1}{9} + \frac{1}{9}$$

$$= \frac{6}{9}$$

$$\|v_2\|^2 = \frac{2}{3}$$

$$\|v_2\| = \sqrt{\frac{2}{3}}$$

$$v_3 = u_3 - \frac{(u_3, v_1)}{\|v_1\|^2} v_1 - \frac{(u_3, v_2)}{\|v_2\|^2} v_2$$

$$= (0, 0, 1) - \frac{1}{2} (1, 1, 1) - \frac{\frac{1}{3}}{\frac{2}{3}} (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$$

$$= (0, 0, 1) + (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) - \frac{1}{2} \cdot \frac{3}{2} [-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}]$$

$$= (0, 0, 1) + (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) - \frac{1}{2} [-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}]$$

$$= (0, 0, 1) + (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) + [\frac{1}{3}, -\frac{1}{6}, -\frac{1}{6}]$$

$$= (0, 0, 1) + (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) + (\frac{1}{3}, -\frac{1}{6}, -\frac{1}{6})$$

$$= (0 - \frac{1}{2} + \frac{1}{3}, 0 - \frac{1}{2} - \frac{1}{6}, 1 - \frac{1}{2} - \frac{1}{6})$$

$$= (0, -\frac{2-1}{6}, \frac{6-2-1}{6})$$

$$= (0, -\frac{1}{6}, \frac{3}{6})$$

$$\begin{aligned}
 v_3 &= (0, -1/2, 1/2) \\
 \|v_3\|^2 &= (v_3, v_3) = (0, -1/2, 1/2) (0, -1/2, 1/2) \\
 &= 0 + \frac{1}{4} + \frac{1}{4} \\
 &= \frac{2}{4} \\
 \|v_3\|^2 &= \frac{1}{2} \\
 \|v_3\| &= \frac{1}{\sqrt{2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{orthogonal set} &= (v_1, v_2, v_3) \\
 &= \left\{ (1, 1, 1), \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right), (0, -1/2, 1/2) \right\}
 \end{aligned}$$

$$\begin{aligned}
 \text{orthogonal Basis} &= \left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right\} \\
 &= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right), \left(\frac{-2/3}{\sqrt{2/3}}, \frac{1/3}{\sqrt{2/3}}, \frac{1/3}{\sqrt{2/3}} \right), \\
 &\quad \left(0, \frac{-1/2}{1/\sqrt{2}}, \frac{1/2}{1/\sqrt{2}} \right) \}
 \end{aligned}$$

② use Gram - Schmidt process find an orthonormal basis for $(1, 0, 1), (0, 1, 1), (1, 3, 3)$

sol

$$\text{Given } u_1 = (1, 0, 1), u_2 = (0, 1, 1), u_3 = (1, 3, 3)$$

$$\text{let } v_1 = u_1 = (1, 0, 1)$$

$$\|v_1\|^2 = (v_1, v_1) = (1, 0, 1) (1, 0, 1)$$

$$= 1 + 0 + 1$$

$$\|v_1\|^2 = 2$$

$$\|v_1\| = \sqrt{2}$$

$$(u_2, v_1) = (0, 1, 1) \cdot (1, 0, 1) = 0 + 0 + 1 = 1$$

$$(u_3, v_1) = (1, 3, 3) (1, 0, 1) = 1 + 0 + 3 = 4$$

$$(u_3, v_1) = 1$$

$$v_2 = u_2 - \frac{(u_2, v_1)}{\|v_1\|^2} v_1$$

$$v_2 = (0, 1, 1) - \frac{1}{2} (1, 0, 1)$$

$$= (0, 1, 1) + (-1/2, 0, -1/2)$$

$$= (0 - 1/2, 1 + 0, 1 - 1/2)$$

$$v_2 = (-1/2, 1, 1/2)$$

$$\|v_2\|^2 = (v_2, v_2) = (-1/2, 1, 1/2) (-1/2, 1, 1/2)$$

$$= \frac{1}{4} + 1 + \frac{1}{4}$$

$$= \frac{1+4+1}{4} = \frac{6}{4}$$

$$\|v_2\|^2 = 3/2$$

$$\|v_2\| = \sqrt{3/2}$$

$$(u_3, v_2) = (1, 3, 3) (-1/2, 1, 1/2)$$

$$= -1/2 + 3 + 3/2$$

$$= \frac{-1+6+3}{2}$$

$$= \frac{8}{2}$$

$$(u_3, v_2) = 4$$

$$v_3 = u_3 - \frac{(u_3, v_1)}{\|v_1\|^2} v_1 - \frac{(u_3, v_2)}{\|v_2\|^2} v_2$$

$$= (1, 3, 3) - \frac{4}{2} (1, 0, 1) - \frac{4}{3/2} (-1/2, 1, 1/2)$$

$$= (1, 3, 3) + (-2, 0, -2) - 4 \times \frac{2}{3} (-1/2, 1, 1/2)$$

$$= (1, 3, 3) + (-2, 0, -2) + (\frac{8}{3}, -\frac{8}{3}, -\frac{8}{3})$$

$$= (1, 3, 3) + (-2, 0, -2) + (\frac{8}{3}, -\frac{8}{3}, -\frac{8}{3})$$

$$= (1, 3, 3) + (-2, 0, -2) + (\frac{8}{3}, -\frac{8}{3}, -\frac{8}{3})$$

$$= \left(1 - 2 + \frac{4}{3}\right), \left(3 + 0 - \frac{8}{3}\right), \left(3 - 2 - \frac{4}{3}\right)$$

$$= \frac{3 - 6 + 4}{3}, \left(\frac{9 - 8}{3}\right), \left(\frac{9 - 6 - 4}{3}\right)$$

$$v_3 = \left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right)$$

$$\|v_3\|^2 = (v_3, v_3) = \left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right) \left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right)$$

$$= \frac{1}{9} + \frac{1}{9} + \frac{1}{9}$$

$$= \frac{3}{9}$$

$$\|v_3\|^2 = \frac{1}{3}$$

$$\|v_3\| = \frac{1}{\sqrt{3}}$$

$$\text{orthogonal set} = (v_1, v_2, v_3)$$

$$= \left\{ (1, 0, 1), \left(-\frac{1}{6}, 1, \frac{1}{2}\right), \left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right) \right\}$$

$$\text{orthonormal Basis} = \left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right\}$$

$$= \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right), \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right),$$

$$\left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3} \right)$$

③. Apply the Gram-Schmidt process to the

$$\text{vectors } u_1 = (1, 0, 1), u_2 = (1, 0, -1), u_3 = (0, 3, 4)$$

to obtain an orthonormal basis for $P^1(\mathbb{R})$

with the standard inner product.

3d. Given

$$u_1 = (1, 0, 1), u_2 = (1, 0, -1), u_3 = (0, 3, 1)$$

Let

$$v_1 = u_1$$

$$v_1 = u_1 = (1, 0, 1)$$

$$\|v_1\|^2 = (v_1, v_1) = (1, 0, 1) \cdot (1, 0, 1)$$

$$= 1 + 0 + 1$$

$$\|v_1\|^2 = 2$$

$$\|v_1\| = \sqrt{2}$$

$$v_2 = u_2 - \frac{(u_2, v_1)}{\|v_1\|^2} v_1$$

$$(u_2, v_1) = (1, 0, -1) \cdot (1, 0, 1)$$

$$= 1 + 0 - 1$$

$$= 0$$

$$= (1, 0, -1) - \frac{0}{2} (1, 0, 1)$$

$$= (1, 0, -1)$$

$$v_2 = (1, 0, -1)$$

$$(u_3, v_1) = (0, 3, 1) \cdot (1, 0, 1)$$

$$= 0 + 0 + 1$$

$$(u_3, v_1) = 1$$

$$\|v_2\|^2 = (v_2, v_2) = (1, 0, -1) \cdot (1, 0, -1)$$

$$= 1 + 0 + 1$$

$$\|v_2\|^2 = 2$$

$$\|v_2\| = \sqrt{2}$$

$$(u_3, v_2) =$$

$$(u_3, v_2) = (0, 3, 1) \cdot (1, 0, -1)$$

$$= 0 + 0 - 1$$

$$(u_3, v_2) = -1$$

$$v_3 = u_3 - \frac{(u_3, v_1)}{\|v_1\|^2} v_1 - \frac{(u_3, v_2)}{\|v_2\|^2} v_2$$

$$= (0, 3, 1) - \frac{1}{2} (1, 0, 1) - \frac{-1}{2} (1, 0, -1)$$

$$= (0, 3, 1) + (-\frac{1}{2}, 0, -\frac{1}{2}) + (\frac{1}{2}, 0, \frac{1}{2})$$

$$= (0 - \frac{1}{2} + \frac{1}{2}, (3 + 0 + 0), (1 - \frac{1}{2} + \frac{1}{2}))$$

$$v_3 = (0, 3, 0)$$

$$\begin{aligned}\|v_3\|^2 &= (v_3, v_3) \\ &= (0, 3, 0) \cdot (0, 3, 0) \\ &= 0 + 9 + 0\end{aligned}$$

$$\|v_3\|^2 = 9$$

$$\|v_3\| = 3$$

$$\begin{aligned}\text{orthogonal set} &= \{v_1, v_2, v_3\} \\ &= \{(1, 0, 1), (1, 0, -1), (0, 3, 0)\}.\end{aligned}$$

$$\begin{aligned}\text{orthonormal set} &= \left(\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right) \\ &= \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right), \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right), \left(0, 1, 0 \right)\end{aligned}$$

$$\text{orthonormal set} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right), \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right), (0, 1, 0)$$

H.W.

①. Apply The Gram-Schmidt process.

$$\nearrow v_1 = (1, 1, 0), v_2 = (2, 1, 3), v_3 = (1, 1, 1)$$

$$\text{Ans: orthogonal set} = \left\{ (1, 1, 0), \left(\frac{1}{2}, -\frac{1}{2}, 3 \right), \left(-\frac{3}{19}, \frac{3}{19}, \frac{4}{19} \right) \right\}$$

$$\text{orthonormal basis set} = \left\{ \frac{1}{\sqrt{2}}(1, 1, 0), \frac{1}{38}(1, -1, 6), \frac{1}{\sqrt{19}}(-3, 3, 4) \right\}$$

②. Apply The Gram-Schmidt process.

$$\nearrow v_1 = (3, 0, 1), v_2 = (-1, 0, 7), v_3 = (2, 9, 11)$$

$$\text{Ans: orthogonal set} = \{(3, 0, 1), (-1, 0, 3), (0, 9, 0)\}$$

$$\text{orthonormal basis set} = \left\{ \frac{1}{5}(3, 0, 1), \frac{1}{5}(-1, 0, 3), (0, 1, 0) \right\}$$

③ Apply The Gram-Schmidt process.

$$u_1 = (1, 1, 1), u_2 = (1, 1, 0), u_3 = (1, 0, 0)$$

$$\text{Ans: orthogonal set} = \left\{ (1, 1, 1), \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right), \left(\frac{2}{3}, -\frac{1}{3}, 0 \right) \right\}$$

$$\text{orthonormal basis set} = \left\{ \frac{1}{\sqrt{3}}(1, 1, 1), \frac{1}{\sqrt{6}}(1, 1, -1), \frac{1}{\sqrt{6}}(2, -1, 0) \right\}$$

④. Apply Gram-Schmidt process to transform the basis vectors $u_1 = (1, 1, 1)$, $u_2 = (1, 1, 0)$, $u_3 = (1, 0, 0)$ of $\mathbb{R}^3$ with the inner product $\langle u, v \rangle = u_1 v_1 + 2u_2 v_2 + 3u_3 v_3$

Sol

$$u_1 = (1, 1, 1), u_2 = (1, 1, 0), u_3 = (1, 0, 0)$$

$$\langle u, v \rangle = u_1 v_1 + 2u_2 v_2 + 3u_3 v_3$$

$$v_1 = u_1 = (1, 1, 1)$$

$$\begin{aligned} \|v_1\|^2 &= (v_1, v_1) = (1, 1, 1)(1, 1, 1) \\ &= 1 + 2(1) + 3(1) \end{aligned}$$

$$\|v_1\|^2 = 6$$

$$\|v_1\| = \sqrt{6}$$

$$v_2 = u_2 - \frac{(u_2, v_1)}{\|v_1\|^2} v_1$$

$$(u_2, v_1) = (1, 1, 0)(1, 1, 1)$$

$$= 1 + 2(1) + 3(0)$$

$$(u_2, v_1) = 3$$

$$= (1, 1, 0) - \frac{3}{6}(1, 1, 1)$$

$$= (1, 1, 0) + (-1/2, -1/2, -1/2) \quad (u_3, v_1) = (1, 0, 0)(1, 1, 1)$$

$$= 1 + 2(0) + 3(0)$$

$$(u_3, v_1) = 1$$

$$= (1 - 1/2, 1 - 1/2, 0 - 1/2)$$

$$v_2 = (1/2, 1/2, -1/2)$$

$$\|v_2\|^2 = (v_2, v_2) = (1/2, 1/2, -1/2)(1/2, 1/2, -1/2)$$

$$= \frac{1}{4} + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{4}\right)$$

$$= \frac{1}{4} + \frac{2}{4} + \frac{3}{4}$$

$$= \frac{6}{4}$$

$$\|v_2\|^2 = \frac{3}{2}$$

$$\|v_2\| = \sqrt{3/2}$$

$$v_3 = u_3 - \frac{(u_3, v_1)}{\|v_1\|^2} v_1 - \frac{(u_3, v_2)}{\|v_2\|^2} v_2$$

$$(u_3, v_2) = (1, 0, 0) \cdot \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$$

$$= \frac{1}{2} + 0 + 0$$

$$(u_3, v_2) = \frac{1}{2}$$

$$v_3 = (1, 0, 0) - \frac{1}{\sqrt{6}} (1, 1, 1) - \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$$

$$= (1, 0, 0) - \frac{1}{\sqrt{6}} (1, 1, 1) - \frac{1}{2}$$

$$v_3 = (1, 0, 0) - \frac{1}{6} (1, 1, 1) - \frac{\frac{1}{2}}{\frac{3}{2}} \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$$

$$= (1, 0, 0) + \left(-\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}\right) - \frac{1}{2} \times \frac{2}{2} \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$$

$$= (1, 0, 0) + \left(-\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}\right) - \frac{1}{2} \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$$

$$= (1, 0, 0) + \left(-\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}\right) + \left(-\frac{1}{6}, -\frac{1}{6}, +\frac{1}{6}\right)$$

$$= \left(1 - \frac{1}{6} - \frac{1}{6}\right), \left(0 - \frac{1}{6} - \frac{1}{6}\right), \left(0 - \frac{1}{6} + \frac{1}{6}\right)$$

$$= \left(\frac{2}{3}, -\frac{1}{3}, 0\right)$$

$$= \frac{4}{9}, -\frac{2}{9}, 0$$

$$\frac{\sqrt{3}}{\sqrt{2}} \left(\frac{2}{3}, -\frac{1}{3}, 0\right)$$

$$v_3 = \left(\frac{2}{3}, -\frac{1}{3}, 0\right) \quad \frac{\sqrt{3} \cdot 2}{\sqrt{2} \cdot \sqrt{3}} \cdot \frac{2}{\sqrt{6}} = \frac{1}{\sqrt{6}}, 0$$

$$\|v_3\|^2 = (v_3, v_3) = \left(\frac{2}{3}, -\frac{1}{3}, 0\right) \cdot \left(\frac{2}{3}, -\frac{1}{3}, 0\right)$$

$$\|v_3\|^2 = \frac{2}{3}$$

$$= \frac{4}{9} + 2\left(\frac{1}{9}\right) + 0$$

$$= \frac{4}{9} + \frac{2}{9} \Rightarrow \frac{6}{9} = \frac{2}{3}$$

$$\|v_3\| = \sqrt{\frac{2}{3}}$$

$$\text{orthogonal set} = (v_1, v_2, v_3) \\ = (1, 1, 1), (1/2, 1/2, -1/2), (2/3, -1/3, 0)$$

$$\text{orthonormal set} = \left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right\}$$

$$= \left\{ \frac{1, 1, 1}{\sqrt{6}}, \frac{1/2, 1/2, -1/2}{\sqrt{3/2}}, \frac{2/3, -1/3, 0}{\sqrt{4/3}} \right\}$$

$$= \left\{ \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right), \left(\frac{\sqrt{2}}{\sqrt{3}} \left(\frac{1}{2} \right), \frac{\sqrt{2}}{\sqrt{3}} \left(\frac{1}{2} \right), \frac{\sqrt{2}}{\sqrt{3}} \left(-\frac{1}{2} \right) \right), \left(\frac{\sqrt{3}}{\sqrt{2}} \left(\frac{2}{3} \right), \frac{\sqrt{3}}{\sqrt{2}} \left(-\frac{1}{3} \right), 0 \right) \right\}$$

$$= \left\{ \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right), \left(\frac{\sqrt{2}}{\sqrt{3}} \frac{1}{\sqrt{2}\sqrt{6}}, \frac{\sqrt{2}}{\sqrt{3}} \frac{1}{\sqrt{2}\sqrt{6}}, \frac{\sqrt{2}}{\sqrt{3}} \frac{-1}{\sqrt{2}\sqrt{6}} \right), \left(\frac{\sqrt{3}}{\sqrt{2}} \frac{2}{\sqrt{3}\sqrt{3}}, \frac{\sqrt{3}}{\sqrt{2}} \frac{-1}{\sqrt{3}\sqrt{3}}, 0 \right) \right\}$$

$$= \left\{ \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right), \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \right), \left(\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, 0 \right) \right\}$$

Norm or length of a vector

Let V be an inner product space and let $x \in V$.
The norm or length of x , denoted by $\|x\|$, is defined as

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

Cauchy - Schwartz Inequality And Triangle Inequality.

Theorem: $\vdash$

- Let V be an inner product space over F then for all $x, y \in V$ & $c \in F$ the following statements are true.
- a) $\|cx\| = |c| \|x\|$
 - b) $\|x\| = 0$ iff $x = 0$
 - c) $|\langle x, y \rangle| \leq \|x\| \|y\|$ (Cauchy - Schwartz Inequality)
 - d) $\|x + y\| \leq \|x\| + \|y\|$ (Triangle Inequality).

Proof

By the definition $\|x\| = \sqrt{\langle x, x \rangle}$

a) To prove $\|cx\| = |c| \|x\|$

$$\begin{aligned}\|cx\| &= \sqrt{\langle cx, cx \rangle} \\ \|cx\|^2 &= \langle cx, cx \rangle \\ &= c \langle x, cx \rangle \\ &= c \bar{c} \langle x, x \rangle \\ \|cx\|^2 &= |c|^2 \|x\|^2 \\ \|cx\| &= |c| \|x\|\end{aligned}$$

b) To prove $\|x\| = 0$ iff $x = 0$

$$\begin{aligned}\|x\| &= 0 \\ \sqrt{\langle x, x \rangle} &= 0 \\ \langle x, x \rangle &= 0 \\ x &= 0\end{aligned}$$

{ Since $\langle x, x \rangle = 0$ iff $x = 0$ }

$\square$ To prove $|\langle x, y \rangle| \leq \|x\| \|y\|$

This inequality is trivially true when $x=0$ or $y=0$
assume $x \neq 0$ & $y \neq 0$

for any $c \in F$,

$$\begin{aligned}\|x - cy\|^2 &\geq 0 \\ 0 &\leq \|x - cy\|^2 \\ &= \langle x - cy, x - cy \rangle\end{aligned}$$

$$= \langle x, x - cy \rangle + \langle -cy, x - cy \rangle$$

$$= \langle x, x \rangle + \langle x, -cy \rangle + \langle cy, cy \rangle + \langle -cy, x \rangle$$

$$= \langle x, x \rangle - \bar{c} \langle x, y \rangle + c \langle y, y \rangle - c \langle y, x \rangle$$

$$\text{let } c = \frac{\langle x, y \rangle}{\langle y, y \rangle}, \quad \bar{c} = \frac{\overline{\langle x, y \rangle}}{\langle y, y \rangle} = \frac{\langle y, x \rangle}{\langle y, y \rangle}$$

$$0 \leq \langle x, x \rangle - \frac{\langle y, x \rangle}{\langle y, y \rangle} \langle x, y \rangle + \frac{\langle x, y \rangle}{\langle y, y \rangle} \frac{\langle y, x \rangle}{\langle y, y \rangle} \langle y, y \rangle$$

$$0 \leq \|x\|^2 - \frac{\overline{\langle x, y \rangle} \langle x, y \rangle}{\|y\|^2} \quad - \frac{\langle x, y \rangle}{\langle y, y \rangle} \frac{\langle y, x \rangle}{\langle y, y \rangle} \langle y, y \rangle$$

$$(\overline{\langle x, y \rangle}) = \langle y, x \rangle$$

$$0 \leq \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2$$

$$|\langle x, y \rangle|^2 \leq \|x\|^2 \|y\|^2 \Rightarrow |\langle x, y \rangle| \leq \|x\| \|y\|$$

do to prove $\|x+y\| \leq \|x\| + \|y\|$.

$$\|x+y\|^2 = \langle x+y, x+y \rangle$$

$$= \langle x, x+y \rangle + \langle y, x+y \rangle$$

$$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$$

$$= \|x\|^2 + \langle x, y \rangle + \overline{\langle x, y \rangle} + \|y\|^2$$

$$= \|x\|^2 + \|y\|^2 + 2 \operatorname{Re} \langle x, y \rangle$$

$$\leq \|x\|^2 + \|y\|^2 + 2 |\langle x, y \rangle|$$

$$\leq \|x\|^2 + \|y\|^2 + 2 \|x\| \|y\|$$

$$\|x+y\|^2 \leq (\|x\| + \|y\|)^2$$

$$\|x+y\| \leq \|x\| + \|y\|$$

① Let $V = C([0,1])$. Define $\|f\| = \int_0^1 |f(t)| dt$ for all $f \in V$. Prove that $\| \cdot \|$ is a norm on the given vector space V .

Sol

$$V = C([0,1])$$

$$\|f\| = \int_0^1 |f(t)| dt \quad \rightarrow \text{①}$$

i] $\|f\| \geq 0$ and $\|f\| = 0 \iff f=0$.

Assume $\|f\| = 0$

$$\int_0^1 |f(t)| dt = 0$$

$$f(t) = 0$$

conversely $f(t) = 0$

$$\begin{aligned} \|f\| &= \int_0^1 f(t) dt \\ &= \int_0^1 0 \cdot dt. \end{aligned}$$

$$\|f\| = 0$$

$$ii] \|af(t)\| = |a| \|f(t)\|$$

$$\begin{aligned} \|af(t)\| &= \int_0^1 |af(t)| dt = \int_0^1 |a| |f(t)| dt \\ &= |a| \int_0^1 |f(t)| dt \\ &= |a| \|f\| \end{aligned}$$

$$iii] \|f+g\| \leq \|f\| + \|g\|$$

$$\begin{aligned} \|f+g\| &= \int_0^1 |(f+g)(t)| dt = \int_0^1 |f(t)+g(t)| dt \\ &\leq \int_0^1 [|f(t)| + |g(t)|] dt = \int_0^1 |f(t)| dt + \int_0^1 |g(t)| dt \\ \|f+g\| &= \|f\| + \|g\| \\ \|f+g\| &\leq \|f\| + \|g\| \end{aligned}$$

$\therefore$ All the axioms of the norm are satisfied.
 $\|f\| = \int_0^1 |f(t)| dt$ is a norm on the given vector space V .

③. In $C([0,1])$, let $f(t) = t$ and $g(t) = e^t$ verify Cauchy-Schwarz and Triangle Inequality.

Sol $f(t) = t$ & $g(t) = e^t \in C[0,1]$

a) Cauchy-Schwarz Inequality states that

$$|\langle f, g \rangle| \leq \|f\| \cdot \|g\| \rightarrow \text{①}$$

$$\langle f, g \rangle = \int_0^1 f(t)g(t) dt$$

Beppo Levi's

$$\begin{aligned} uv &\rightarrow y du \\ u=t &\rightarrow y=e^t \\ du=dt &\quad v=e^t \\ &\quad t e^t - \int e^t dt \\ &\quad = t e^t - e^t \end{aligned}$$

$$\langle f, g \rangle = \int_0^1 t e^t dt = \left[t e^t - e^t \right]_0^1 = (e - e) - (-1) = 0 + 1 = 1$$

$$\langle f, g \rangle = 1 \rightarrow \text{②}$$

$$\text{Now } \|f\|^2 = \langle f, f \rangle = \int_0^1 [f(t)]^2 dt$$

$$= \int_0^1 t^2 dt = \left[\frac{t^3}{3} \right]_0^1 = \frac{1}{3}$$

$$\|f\|^2 = \frac{1}{3}$$

$$\|f\| = \frac{1}{\sqrt{3}}$$

$$\|g\|^2 = \langle g, g \rangle = \int_0^1 (g(t))^2 dt = \int_0^1 (e^t)^2 dt$$

$$\|f\| \cdot \|g\| = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{e^2-1}}{\sqrt{2}}$$

$$= \frac{\sqrt{e^2-1}}{\sqrt{6}}$$

$$= \sqrt{\frac{1}{6}} = \frac{2.52}{2.44}$$

$$1 \leq 1.031$$

$$|\langle f, g \rangle| \leq \|f\| \cdot \|g\|$$

$\therefore$ Cauchy-Schwarz Inequality is verified.

$$= \int_0^1 e^{2t} dt = \left[\frac{e^{2t}}{2} \right]_0^1$$

$$\|g\|^2 = \frac{e^2}{2} - \frac{1}{2}$$

$$\|g\|^2 = \frac{e^2-1}{2}$$

$$\|g\| = \sqrt{\frac{e^2-1}{2}}$$

b) Triangle Inequality states that

$$\|f+g\| \leq \|f\| + \|g\|$$

$$\|f+g\|^2 = \langle f+g, f+g \rangle = \int_0^1 [f(t) + g(t)]^2 dt$$

$$= \int_0^1 (t + e^t)^2 dt$$

$$= \int_0^1 t^2 + e^{2t} + 2te^t dt$$

$$= \int_0^1 t^2 dt + \int_0^1 e^{2t} dt + 2 \int_0^1 te^t dt$$

$$= \left[\frac{t^3}{3} \right]_0^1 + \left[\frac{e^{2t}}{2} \right]_0^1 + 2 \left[te^t - e^t \right]_0^1$$

$$= \frac{1}{3} + \frac{e^2}{2} - \frac{1}{2} + 2[(e+e) - (1)]$$

$$= \frac{1}{3} + \frac{e^2}{2} - \frac{1}{2} + 2(e+e-1)$$

$$= \frac{1}{3} + \frac{e^2}{2} - \frac{1}{2} + 2$$

$$= \frac{2+3e^2-2}{6}$$

$$= \frac{1+3e^2}{6}$$

$$\|f+g\|^2 = 5.5273$$

$$\|f+g\| = \sqrt{5.5273}$$

$$\|f+g\| = 2.3511$$

$$\|f\| + \|g\| = \frac{1}{\sqrt{2}} + \frac{\sqrt{e^2-1}}{e} = 0.7071 + 1.7873$$

$$= 2.4944$$

$$\|f\| + \|g\| = 2.4944$$

$$\|f+g\| \leq \|f\| + \|g\|$$

Triangle inequality is Verified

$$e^2 = 7.389$$

$$3e^2 = 22.167 + 11$$

$$= 33.167$$

$$= \sqrt{5.5273}$$

$$0.7071 +$$

$$1.7873$$

Unit - II

Linear Transformation And Diagonalization

Linear transformation

Let V & W be vector spaces F . A function $T: V \rightarrow W$ is a linear transformation from V to W if for all $x, y \in V$ & $c \in F$.

$$i] T(x+y) = T(x) + T(y) \quad \text{Additivity}$$

$$ii] T(cx) = cT(x) \quad [\text{scalar multiplication}]$$

Q. Show that T is linear when $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(a_1, a_2) = (2a_1 + a_2, a_1)$

so) let $x = (a_1, a_2)$ & $y = (b_1, b_2)$

i] Additivity:

$$x+y = (a_1, a_2) + (b_1, b_2)$$

$$x+y = (a_1+b_1, a_2+b_2)$$

$$T(x+y) = T(a_1+b_1, a_2+b_2) \quad \because T(a_1, a_2) = (2a_1 + a_2, a_1)$$
$$= [2(a_1+b_1) + (a_2+b_2), a_1+b_1]$$

$$= [2a_1 + 2b_1 + a_2 + b_2, a_1 + b_1]$$

$$= (2a_1 + a_2, a_1) + (2b_1 + b_2, b_1) \rightarrow \text{QED}$$

$$T(x+y) = T(a_1, a_2) + T(b_1, b_2)$$

ii] scalar multiplication:-

$$\text{let } c \in \mathbb{R} \text{ & } x = (a_1, a_2) \in \mathbb{R}^2$$

$$x = (a_1, a_2)$$

$$cx = c(a_1, a_2)$$

$$cx = (ca_1, ca_2)$$

$$\therefore T(ca_1, ca_2) = (2ca_1 + ca_2, ca_1)$$

$$T(ca_1, ca_2) = (2(ca_1) + ca_2, ca_1)$$

$$= c(2a_1 + a_2, a_1)$$

$$= cT(a_1, a_2) \rightarrow \textcircled{2}$$

$$\textcircled{1} = \textcircled{2}$$

T is linear.

② Show that T is not linear when $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which is defined by $T(a_1, a_2) = (\sin a_1, 0)$

Sol

Given $T(a_1, a_2) = (\sin a_1, 0)$

$$T(cx) = cT(x) \quad \& \quad T(x+y) = T(x) + T(y)$$

$$\text{Let } T\left(\frac{2\pi}{2}, 0\right) = T(\pi, 0)$$

$$= (\sin \pi, 0)$$

$$= (0, 0) \rightarrow \textcircled{1}$$

$$\sin \pi = 0$$

$$\sin \frac{\pi}{2} = 1$$

$$\text{Let } T\left(\frac{2\pi}{2}, 0\right) = 2T\left(\frac{\pi}{2}, 0\right)$$

$$= 2T\left(\frac{\pi}{2}, 0\right)$$

$$= 2(1, 0)$$

$$= (2, 0) \rightarrow \textcircled{2}$$

$$\textcircled{1} \neq \textcircled{2}$$

T is not linear.

③ Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ which is defined by $T(1, 0, 3) = (1, 1)$ &

$T(-2, 0, -6) = (2, 1)$ Verify which T is a linear transformation or not?

Sol

Given $T(1, 0, 3) = (1, 1)$

$T(-2, 0, -6) = (2, 1)$

$\rightarrow \textcircled{1}$

formula

$$T(x+y) = T(x) + T(y)$$

$$T(cx) = cT(x)$$

$$\text{let } T(-2, 0, -6) = T(-2(1, 0, 3))$$

$$= -2T(1, 0, 3)$$

$$= -2(1, 1) \text{ by } \textcircled{1}$$

$$T(-2, 0, -6) = (-2, -2) \rightarrow \textcircled{2}$$

$$\text{from } \textcircled{1} \quad T(-2, 0, -6) = (2, 1) \rightarrow \textcircled{3}$$

$$\textcircled{2} \neq \textcircled{3} \quad \therefore T \text{ is not linear}$$

Theorem $\left(\begin{smallmatrix} + \\ - \end{smallmatrix} \right)$

state and prove dimension theorem (or)

Rank - nullity Theorem.

statement

let V & W be vector spaces and

let $T: V \rightarrow W$ be linear

If V is finite dimension, then $\text{nullity}(T) +$

$$\text{rank}(T) = \dim(V)$$

proof

$$\text{let } \dim(V) = n \text{ \& } \dim(N(T)) = k \rightarrow \textcircled{A}$$

$\{v_1, v_2, \dots, v_k\}$ is a basis for $N(T)$ &

$B = \{v_1, v_2, \dots, v_n\}$ for V .

$\therefore S = \{T(v_{k+1}), T(v_{k+2}), \dots, T(v_n)\}$ is a basis for $R(T)$.

$$\therefore R(T) = \text{span}(T(B))$$

$$= \text{span}(T(v_1), T(v_2), \dots, T(v_n))$$

$$= \text{span}(T(v_{k+1}), T(v_{k+2}), \dots, T(v_n))$$

$$R(T) = \text{span}(S) \rightarrow \textcircled{B}$$

prove that S is a linearly independent set.

$$\text{Let } b_{k+1}T(v_{k+1}) + b_{k+2}T(v_{k+2}) + \dots + b_n T(v_n) = 0$$

$$T(b_{k+1}v_{k+1}) + T(b_{k+2}v_{k+2}) + \dots + T(b_nv_n) = 0$$

$$T(b_{k+1}v_{k+1} + b_{k+2}v_{k+2} + \dots + b_nv_n) = 0$$

$$\therefore N(T) = \{x \in V; T(x) = 0\}$$

$$b_{k+1}v_{k+1} + b_{k+2}v_{k+2} + \dots + b_nv_n = 0 \in N(T)$$

$\therefore v_1, v_2, \dots, v_k$ is a basis for $N(T)$.

$$b_{k+1}v_{k+1} + b_{k+2}v_{k+2} + \dots + b_nv_n =$$

$$b_1v_1 + b_2v_2 + \dots + b_kv_k$$

$$(-b_1)v_1 + (-b_2)v_2 + \dots + (-b_k)v_k +$$

$$b_{k+1}v_{k+1} + b_{k+2}v_{k+2} + \dots + b_nv_n = 0$$

$$-b_1 = -b_2 = \dots = -b_n = 0$$

$$b_1 = b_2 = \dots = b_n = 0$$

$\{v_1, v_2, \dots, v_n\}$ is linearly independent

$\therefore$ is a linearly independent $\rightarrow \textcircled{2}$

From $\textcircled{1}$ & $\textcircled{2}$

S is the basis for $R(T)$

$$\dim(R(T)) = n - k$$

$$\dim(R(T)) = \dim(V) - \dim(N(T)) \text{ by } \textcircled{1}$$

$$\dim(R(T)) + \dim(N(T)) = \dim(V)$$

(or)

$$\dim(V) = \text{rank}(T) + \text{nullity}(T)$$

Hence the proof.

①. Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$.
Find Nullity (T).

Sol Let $N(T) = \{x \in V, T(x) = 0\} \rightarrow \textcircled{1}$

Let $x \in \mathbb{R}^2$

$$x = (a_1, a_2)$$

$$T(x) = T(a_1, a_2) = 0 \rightarrow \textcircled{2} \text{ by } \textcircled{1}$$

$$T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2) \rightarrow \textcircled{3}$$

$$(a_1 + a_2, 0, 2a_1 - a_2) = (0, 0, 0) \text{ by } \textcircled{3}$$

$$a_1 + a_2 = 0 \rightarrow \textcircled{4}$$

$$2a_1 - a_2 = 0 \rightarrow \textcircled{5}$$

$$\textcircled{4} \times \textcircled{5} \Rightarrow 2a_1 + 2a_2 = 0$$

$$\textcircled{3} \times \textcircled{1} \Rightarrow \begin{array}{r} (-) \\ 2a_1 - a_2 = 0 \end{array}$$

$$\hline 3a_2 = 0$$

$$a_2 = 0$$

Sub $a_2 = 0$ in $\textcircled{4}$

$$a_1 + a_2 = 0$$

$$a_1 = 0$$

Sol

$$a_1 = 0$$

$$a_2 = 0$$

From $\textcircled{1}$

$$N(T) = \{x \in V, T(x) = 0\}$$

$$= \{(a_1, a_2) \in V, T(a_1, a_2) = 0\}$$

$$N(T) = \{(0, 0) \in V, T(0, 0) = (0, 0)\}$$

$$N(T) = \{0\}$$

$$\dim(R(T)) = \text{rank}(T) = 2$$

$$\text{Nullity}(T) = \dim(V) - \text{rank}(T)$$

$$= 2 - 2$$

$$\text{Nullity} = 0$$

②. Let $T: P_2(\mathbb{R}) \rightarrow P_3(\mathbb{R})$. Define
 $T[f(x)] = xf(x) + f'(x)$. Find the basis for
 $R(T)$ & find the value of $R(T)$.

Sol

$$\text{Basis} = \{1, x, x^2\}$$

$x=1$	$x=x$	$x=x^2$
$f(x)=1$	$f(x)=x$	$f(x)=x^2$
$f'(x)=0$	$f'(x)=1$	$f'(x)=2x$

$$T[f(x)] = xf(x) + f'(x)$$

$$T[f(x)] = x \cdot 1 + 0 = x \Rightarrow \textcircled{u}$$

$$T[f(x)] = x \cdot x + 1 = x^2 + 1 \Rightarrow \textcircled{v}$$

$$T[f(x^2)] = x \cdot x^2 + 2x = x^3 + 2x \Rightarrow \textcircled{w}$$

$$au + bv + cw = 0$$

$$ax + bx^2 + b + cx^3 + 2cx = 0$$

$$cx^3 + bx^2 + x(a+2c) + b = 0$$

Equating coefficient both sides

$$c=0, b=0$$

$$a+2c=0$$

$$a+2 \cdot 0 = 0$$

$$\boxed{a=0}$$

Solution

$$a=b=c=0$$

$$\text{Basis} = \{x, x^2+1, x^3+2x\}$$

$$R(T) = 3.$$

③ verify that $T: \mathbb{R}^3 \rightarrow \mathbb{R}$ & $T(u) = \|u\|$ is a
 linear transformation or not.

Sol

$$T(u+v) = T(u) + T(v)$$

$$T(\alpha u) = \alpha T(u).$$

① Additivity

$$\|T(u)\| = \|u\|$$

$$T(u+v) = \|u+v\|$$

$$T(u+v) \leq \|u\| + \|v\| \text{ by Triangle Inequality}$$

$$T(u+v) \neq \|u\| + \|v\|$$

$\therefore$ Additivity fails

② scalar multiplication:-

$$T(\alpha u) = \alpha(T(u)) \rightarrow \textcircled{1}$$

$$T(\alpha u) = \|\alpha u\| = |\alpha| \|u\| \rightarrow \textcircled{2}$$

$$\textcircled{1} \neq \textcircled{2}$$

T is not a linear map

① show that T is linear when $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by $T(a_1, a_2) = (2a_1 + a_2, a_1)$

sol

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ is } T(a_1, a_2) = (2a_1 + a_2, a_1)$$

$$T(x+y) = T(x) + T(y)$$

$$c \in \mathbb{R}, x, y \in \mathbb{R}^2$$

$$x = (a_1, a_2), y = (b_1, b_2)$$

$$x+y = (a_1, a_2) + (b_1, b_2)$$

$$x+y = (a_1+b_1, a_2+b_2)$$

$$c(x+y) = (c(a_1+b_1), c(a_2+b_2))$$

$$T(c(x+y)) = T(\overset{a_1}{c(a_1+b_1)}, \overset{a_2}{c(a_2+b_2)})$$

$$= (2(c(a_1+b_1)) + c(a_2+b_2), c(a_1+b_1))$$

$$= (2ca_1 + 2cb_1 + ca_2 + cb_2, ca_1 + cb_1)$$

$$= 2ca_1 + ca_2, 2cb_1 + cb_2 \rightarrow \textcircled{2}$$

and

$$T(x) + T(y) = T(a_1, a_2) + T(b_1, b_2)$$

$$= (2a_1 + a_2, a_1) + (2b_1 + b_2, b_1)$$

$$c(T(x) + T(y)) = 2ca_1 + ca_2, ca_1 + 2cb_1 + cb_2 \rightarrow \textcircled{3}$$

$$\textcircled{2} = \textcircled{3}$$

T is linear

Q. Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(x, y) = (x+2y, 2x+3y+2)$ check whether T is linear or not.

Sol

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$T(x, y) = (x+2y, 2x+3y+2)$$

$$T(cx+y) = cT(x) + T(y)$$

$$x = (a_1, a_2), \quad y = (b_1, b_2)$$

$$x+y = (a_1+b_1, a_2+b_2)$$

$$cx+y = (ca_1+b_1, ca_2+b_2)$$

$$T(cx+y) = T(ca_1+b_1, ca_2+b_2)$$

$$= (ca_1+b_1+2(ca_2+b_2), 2(ca_1+b_1)+3(ca_2+b_2)+2)$$

$$\begin{aligned} T(cx+y) &= (ca_1+b_1+2ca_2+2b_2, 2ca_1+2b_1+3ca_2+3b_2+2) \\ &= (ca_1+2ca_2, 2ca_1+3ca_2+b_1+2b_2, 2b_1+3b_2+2) \end{aligned}$$

$$cT(x) + T(y) = cT(a_1, a_2) + T(b_1, b_2)$$

$$= c[a_1+2a_2, 2a_1+3a_2+2]$$

$$+ [b_1+2b_2, 2b_1+3b_2+2]$$

$$= (ca_1+2ca_2, 2ca_1+3ca_2+2c+b_1+2b_2, 2b_1+3b_2+2c+2)$$

$$= (ca_1+2ca_2, 2ca_1+3ca_2+c+b_1+2b_2, 2b_1+3b_2+2c+2)$$

$$b_1+2b_2, 2b_1+3b_2+2c+2$$

$$\textcircled{2} \neq \textcircled{3}$$

T is not linear.

43

Null space, Range, Dimension Theorem

Definition: Null space or kernel

Null space or kernel of a linear transformation $T: V \rightarrow W$ is the set of all vectors $x \in V$ such that $T(x) = 0$.
Null space is denoted by $N(T)$.

$$N(T) = \{x \in V \mid T(x) = 0\}$$

Definition: Range or Image

Range or Image of a linear transformation $T: V \rightarrow W$ is the subset of W consisting all images (under T) of vectors in V .

Range is denoted by $R(T)$

$$R(T) = \{T(x) \mid x \in V\}$$

Definition: nullity and Rank of T

Let $T: V \rightarrow W$ be a linear transformation

Then the dimension of $T(V)$ is called the rank of T and the dimension of kernel of T is called the nullity of T .

$$\text{Nullity}(T) = \dim N(T)$$

$$\text{Rank}(T) = \dim R(T)$$

Definition: Identity and zero transformation

Let V and W be vector spaces over F , $I_V: V \rightarrow V$ by $I_V(x) = x$ for all $x \in V$. I_V is called the Identity transformation.

$T_0: V \rightarrow W$ by $T_0(x) = 0$ for all $x \in V$.

T_0 is called the zero transformation

Both I_V and T_0 are linear.

$$N(I_V) = \{0\}, R(I_V) = V, N(T_0) = V, R(T_0) = \{0\}.$$

Definition Theorem: (Rank - Nullity Theorem) (only statement)

Let V and W be vector spaces and $T: V \rightarrow W$ be linear transformation. If V is finite dimensional, then

$$\text{Nullity}(T) + \text{Rank}(T) = \dim(V).$$

Consider $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by
 $T(x, y, z) = (x-y, 2z)$ is linear.
 Find nullity (T) and Rank (T).

Sol

$$T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

$$T(x, y, z) = (x-y, 2z)$$

Find $N(T)$ & $R(T)$

$$N(T) = \{(x, y, z) \in V \mid T(x, y, z) = 0\}$$

$$T(x, y, z) = 0, (x-y, 2z) = 0$$

$$x-y=0, 2z=0$$

$$x=y, z=0$$

$$N(T) = \{(x, x, 0) \mid x \in \mathbb{R}\}$$

$$= \{x(1, 1, 0) \mid x \in \mathbb{R}\}$$

$N(T)$ is spanned by single vector $(1, 1, 0)$

$\{(1, 1, 0)\}$ is a basis of $N(T)$

$$\text{nullity}(T) = \dim N(T) = 1$$

$$\text{nullity}(T) + \text{Rank}(T) = \dim(V)$$

$$1 + \text{Rank}(T) = 3$$

$$\text{Rank}(T) = 2$$

Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ Define $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$. Find the basis for $N(T)$ and evaluate the nullity of T .

Sol $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$

$$T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$$

$$N(T) = \{x \in V \mid T(x) = 0\}$$

$$R(T) = \{T(x) \mid x \in V\}$$

$$T(a_1, a_2, a_3) = 0 \Rightarrow (a_1 - a_2, 2a_3) = 0$$

$$a_1 - a_2, 2a_3 = 0$$

$$a_1 - a_2 = 0, 2a_3 = 0$$

$$a_1 = a_2, a_3 = 0$$

$$N(T) = \{(a_1, a_1, 0) \mid a_1 \in \mathbb{R}\}$$

$$= \{a(1, 1, 0) \mid a \in \mathbb{R}\}$$

$$\text{nullity}(T) = \dim N(T) = 1$$

$$N(T) = \{a(1, 1, 0) \in \mathbb{R}^3\}$$

$\therefore (1, 1, 0)$ is the basis of $N(T)$.

Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$. Define $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$. Find the basis for $N(T)$ & compute $N(T)$

Sol

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$N(T) = \{x \in V \mid T(x) = 0\}$$

Find $N(T)$

$$T(x) = 0$$

$$T(a_1, a_2) = 0$$

$$(a_1 + a_2, 0, 2a_1 - a_2) = 0$$

$$a_1 + a_2 = 0, 2a_1 - a_2 = 0$$

$$a_1 + a_2 = 0 \Rightarrow a_2 = -a_1$$

$a_1 = 0$ sub in $a_2 = -a_1$

$$a_1 + a_2 = 0 \Rightarrow a_2 = -a_1$$

$$a_2 = 0, \text{ then } a_1 = 0$$

$$N(T) = \{(0, 0) \in \mathbb{R}^2\}$$

$N(T)$ is spanned by a zero vector.

Null space of T is zero dimensional

$$N(T) = \{0\}, \text{ nullity}(T) = 0$$

Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is defined by

$$T(x, y, z) = (x-y, 2z)$$

is a linear transformation. Find the basis for $R(T)$ and find the value of $\text{Rank}(T)$

Sol $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2, T(x, y, z) = (x-y, 2z)$

$$R(T) = \{T(x) \mid x \in V\}$$

$$N(T) = \{x \in V \mid T(x) = 0\}$$

$$T(x, y, z) = 0 \Rightarrow (x-y, 2z) = 0$$

$$x-y=0, 2z=0$$

$$(x, y) = (a, b), 2c = 0$$

$$a-b=x$$

$$2c=y$$

$$c = \frac{1}{2}y$$

$$T(a, b, c) = (a-b, 2c) = (x, y)$$

$\{(1, 0), (0, 1)\}$ is a standard basis

It is also a s.b for $R(T)$

Basis of $R(T)$ is $\{(1, 0), (0, 1)\}$ & $\text{Rank}(T) = 2$

Matrix Representation of Linear Transformation

① Find a basis for the range of the matrix $A = \begin{bmatrix} 5 & 2 \\ 1 & 0 \end{bmatrix}$

Sol let $A = \begin{bmatrix} 5 & 2 \\ 1 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} R_2 = 5R_2 - R_1$$

which is echelon form of matrix

Thus the set of non-zero vectors

$\{(1, 2), (0, -2)\}$ is the basis for $R(A)$.

② obtain the matrix representing the linear transformation $T: V_3(\mathbb{R}) \rightarrow V_3(\mathbb{R})$ given by $T(a, b, c) = (3a, a-b, 2a+b+c)$ with respect to the standard basis (e_1, e_2, e_3)

Sol let $e_1 = (1, 0, 0)$

$$e_2 = (0, 1, 0)$$

$$e_3 = (0, 0, 1)$$

$$T(a, b, c) = (3a, a-b, 2a+b+c)$$

$$T(e_1) = T(1, 0, 0) = (3(1), 1-0, (2(1)+0+0))$$

$$T(e_1) = (3, 1, 2)$$

$$T(e_2) = T(0, 1, 0) = (3(0), (0-1), (2(0)+1+0))$$

$$T(e_2) = (0, -1, 1)$$

$$T(e_3) = T(0, 0, 1) = (3(0), 0-0, (2(0)+0+1))$$

$$T(e_3) = (0, 0, 1)$$

$$\text{Matrix} = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

③ Prove that matrix representing the linear transformation the linear transformation $T: V_3(\mathbb{R}) \rightarrow V_3(\mathbb{R})$ given by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$ w.r.to the standard basis (e_1, e_2, e_3) .

Sol

$$\text{let } e_1 = (1, 0, 0)$$

$$e_2 = (0, 1, 0)$$

$$e_3 = (0, 0, 1)$$

$$T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$$

$$T(e_1) = T(1, 0, 0) = (1 - 0, 2(0)) = (1, 0)$$

$$T(e_2) = T(0, 1, 0) = (0 - 1, 2(0)) = (-1, 0)$$

$$T(e_3) = T(0, 0, 1) = (0 - 0, 2(1)) = (0, 2)$$

$$\text{Matrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

④ Find the representing the linear transformation.

$T: V_3(\mathbb{R}) \rightarrow V_3(\mathbb{R})$ given by $T(a, b, c) = (3a, a - b, (2a + b + c))$ with respect to the standard basis $\{e_1, e_2, e_3\}$

Sol

$T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

④ Let B_1 and B_2 be the standard ordered bases for $\mathbb{R}^n$

and $\mathbb{R}^m$, respectively. Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$, compute $[T]_{B_2}^{B_1}$

for $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by $T(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$

Sol

$$\mathbb{R}^2 \Rightarrow e_1 = (1, 0), e_2 = (0, 1)$$

$$\mathbb{R}^3 \Rightarrow e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$$

$$T(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$$

$$T(e_1) = T(1, 0) = (2(1) - 0, 3(1) + 4(0), 1)$$

$$T(e_2) = (2(0) - 1, 3(0) + 4(1), 0)$$

$$T(p_2) = T(0,1) = (2(0)-1, 3(0)+4(1), 0)$$

~~$$= (-1, 4, 0)$$~~

$$T(p_2) = (-1, 4, 0)$$

$$[T]_{B_1}^{B_2} = \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{bmatrix}$$

6. Let $T: M_{2 \times 2} \mathbb{R} \rightarrow P_2(\mathbb{R})$

$$\text{Define: } T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a+b) + 2dx + bx^2$$

$$\text{Let } B_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ \& } B_2 = \{1, x, x^2\},$$

compute $[T]_{B_1}^{B_2}$.

Sol Given that $T: M_{2 \times 2}(\mathbb{R}) \rightarrow P_2(\mathbb{R})$

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a+b) + 2dx + bx^2$$

$$B_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$B_2 = \{1, x, x^2\}$$

$$T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = (1+0) + 2(0)x + 0x^2 = 1 + 0x + 0x^2$$

$$T \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = (0+1) + 2(0)x + 1x^2 = 1 + 0x + x^2$$

$$T \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = (0+0) + 2(1)x + 0x^2 = 0 + 2x + 0x^2$$

$$T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = (0+0) + 2(0)x + 0x^2 = 0 + 0x + 0x^2$$

$$[T]_{B_1}^{B_2} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

⑤. Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be linear defined by

$$T(x, y) = (x+y, 2x-y, 3x+5y). \text{ If } B_1 = \{(1, 1), (0, -1)\} \text{ and}$$

$B_2 = \{(1, 1, 1), (1, 0, 1), (0, 0, 1)\}$ are the bases for $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively, find the matrix representation of T .

sol.

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^3 \text{ is linear}$$

$$T(x, y) = (x+y, 2x-y, 3x+5y)$$

$$B_1 = \{e_1, e_2\} = \{(1, 1), (0, -1)\}$$

$$B_2 = \{e_1, e_2, e_3\} = \{(1, 1, 1), (1, 0, 1), (0, 0, 1)\}$$

$$T(e_1) = T(1, 1) = (1+1, 2(1)-1, 3(1)+5(1))$$

$$T(e_1) = (2, -1, 8)$$

$$T(e_2) = T(0, -1) = (0-1, 2(0)-1, 3(0)+5(-1))$$

$$T(e_2) = (-1, 1, -5)$$

$(2, -1, 8)$ is a linear combination of $(1, 1, 1), (1, 0, 1), (0, 0, 1)$

$$(2, -1, 8) = a(1, 1, 1) + b(1, 0, 1) + c(0, 0, 1)$$

$$(2, -1, 8) = (a+b, a, a+b+c)$$

$$a+b = 2 \rightarrow \textcircled{1}$$

$$a = 1 \rightarrow \textcircled{2}$$

$$a+b+c = 8 \rightarrow \textcircled{3}$$

$$\begin{cases} a=1 & \text{sub } \textcircled{1} \\ b=2-1 & \\ b=1 & \end{cases}$$

$$a=1, b=1 \text{ sub in } \textcircled{3}$$

$$1+1+c=8$$

$$c=8-2$$

$$c=6$$

$$(2, -1, 8) = 1(1, 1, 1) + 1(1, 0, 1) + 6(0, 0, 1)$$

$$(-1, 1, -5) = a(1, 1, 1) + b(1, 0, 1) + c(0, 0, 1)$$

$$(-1, 1, -5) = (a+b, a, a+b+c)$$

$$a+b = -1 \rightarrow \textcircled{4}$$

$$a = 1 \rightarrow \textcircled{5}$$

$$a+b+c = -5 \rightarrow \textcircled{6}$$

$$\begin{cases} a=1 & \text{sub in } \textcircled{4} \\ 1+b=-1 & \\ b=-1-1 & \\ b=-2 & \end{cases}$$

$$a=1, b=-2 \text{ sub in } \textcircled{6}$$

$$1-2+c=-5$$

$$c=-5+1$$

$$c=-4$$

$$\therefore (1, 1, -5) = 1(1, 1, 1) - 2(1, 0, 1) - 4(0, 0, 1)$$

The Matrix representation of T is

$$[T]_{B_2}^{B_1} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 6 & -4 \end{bmatrix}$$

Q) consider the linear transformation $T: V_2(\mathbb{R}) \rightarrow V_2(\mathbb{R})$ given by $T(a, b) = (a, a+b)$. If $\{e_1, e_2\}$ is the basis for domain and $\{(1, 1), (1, -1)\}$ is the basis for the range, find the matrix representation of the matrix.

Sol

Given $T: V_2(\mathbb{R}) \rightarrow V_2(\mathbb{R})$,

$$T(a, b) = (a, a+b)$$

$$B_1 = \{e_1, e_2\} = \{(1, 0), (0, 1)\}$$

$$B_2 = \{e_1, e_2\} = \{(1, 1), (1, -1)\}$$

$$\Rightarrow T(a, b) = (a, a+b)$$

$$T(e_1) = T(1, 0) = (1, 1+0) = (1, 1)$$

$$T(e_2) = T(0, 1) = (0, 0+1) = (0, 1)$$

$(1, 1)$ is a linear combination of $(1, 1)$ and $(1, -1)$

$$(1, 1) = a(1, 1) + b(1, -1)$$

$$(1, 1) = (a+b, a-b)$$

$$a+b = 1 \rightarrow \textcircled{1}$$

$$a-b = 1 \rightarrow \textcircled{2}$$

$$2a = 2$$

$$a = 1$$

$$a=1 \text{ sub in } \textcircled{1}$$

$$a+b = 1$$

$$1+b = 1$$

$$\boxed{b = 0}$$

$$\therefore (1, 1) = 1(1, 1) + 0(1, -1)$$

$$[T]_{B_2}^{B_1} = \begin{bmatrix} 1 & 1/2 \\ 0 & -1/2 \end{bmatrix}$$

$$(0, 1) = a(1, 1) + b(1, -1)$$

$$(0, 1) = (a+b, a-b)$$

$$a+b = 0 \rightarrow \textcircled{3}$$

$$a-b = 1 \rightarrow \textcircled{4}$$

$$2a = 1$$

$$a = 1/2$$

$$a = 1/2 \text{ sub in } \textcircled{3}$$

$$1/2 + b = 0$$

$$\boxed{b = -1/2}$$

$$\therefore (0, 1) = 1/2(1, 1) - 1/2(1, -1)$$

③ Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be linear defined by
 $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let B_1 be the standard
 basis of $\mathbb{R}^2$ and $B_2 = \{(1, 1, 0), (0, 1, 1), (2, 2, 3)\}$ is the
 basis of $\mathbb{R}^3$. Find the matrix form of T .

Sol

$T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is linear

$$B_1 = \{e_1, e_2\} = \{(1, 0), (0, 1)\}$$

$$B_2 = \{e_1, e_2, e_3\} = \{(1, 1, 0), (0, 1, 1), (2, 2, 3)\}$$

$$T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$$

$$T(e_1) = T(1, 0) = (1 - 0, 1, 2(1) + 0) = (1, 1, 2)$$

$$T(e_2) = T(0, 1) = (0 - 1, 0, 2(0) + 1) = (-1, 0, 1)$$

$(1, 1, 2)$ is a linear combination $(1, 1, 0), (0, 1, 1), (2, 2, 3)$.

$$(1, 1, 2) = a(1, 1, 0) + b(0, 1, 1) + c(2, 2, 3)$$

$$(1, 1, 2) = (a + 2c, a + b + 2c, b + 3c)$$

$$a + 2c = 1 \rightarrow \textcircled{1}$$

$$a + b + 2c = 1 \rightarrow \textcircled{2}$$

$$b + 3c = 2 \rightarrow \textcircled{3}$$

$$\begin{array}{rcl} a + 2c & = & 1 \\ a + b + 2c & = & 1 \end{array}$$

$$a + 2c = 1$$

$$a + b + 2c = 1$$

$$0 + b + 0 = 0$$

$$\boxed{b = 0}$$

$b = 0$ sub in $\textcircled{3}$

$$b + 3c = 2$$

$$0 + 3c = 2$$

$$\boxed{c = 2/3}$$

$b = 0$ & $c = 2/3$ sub in $\textcircled{1}$

$$a + 0 + 2 \cdot \frac{2}{3} = 1$$

$$a = 1 - \frac{4}{3}$$

$$\boxed{a = -\frac{1}{3}}$$

$$\therefore (1, 1, 2) = -\frac{1}{3}(1, 1, 0)$$

$$+ 0(0, 1, 1) + \frac{2}{3}(2, 2, 3)$$

$(-1, 0, 1)$ as a linear combination of $(1, 1, 0), (0, 1, 1), (2, 2, 3)$

$$(-1, 0, 1) = a(1, 1, 0) + b(0, 1, 1) + c(2, 2, 3)$$

$$(-1, 0, 1) = (a+2c, a+b+2c, b+3c)$$

$$a+2c = -1 \rightarrow \textcircled{A}$$

$$a+b+2c = 0 \rightarrow \textcircled{B}$$

$$b+3c = 1 \rightarrow \textcircled{C}$$

$\textcircled{A} \& \textcircled{B}$

$$\begin{array}{r} a+2c = -1 \\ a+b+2c = 0 \end{array}$$

$$-b = -1$$

$$b = +1$$

$b=1$ sub in $\textcircled{C}$

$$1+3c = 1$$

$$3c = 0$$

$$3c = 0$$

$$\boxed{c = 0}$$

$b=1$ & $c=0$ sub in $\textcircled{A}$

$$a+b+2c = 0$$

$$a+1+0 = 0$$

$$\boxed{a = -1}$$

$$\therefore (-1, 0, 1) = -1(1, 1, 0) + 1(0, 1, 1) + 0(2, 2, 3)$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{B_1}^{B_2} = \begin{bmatrix} -1/3 & -1 \\ 0 & 1 \\ 2/3 & 0 \end{bmatrix} //$$

① Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the linear transformation defined by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{pmatrix} \text{ find the matrix for the}$$

transformation T with respect to the bases $B = \{u_1, u_2\}$

for $\mathbb{R}^2$ and $B' = \{v_1, v_2, v_3\}$ for $\mathbb{R}^3$ where $u_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$,

$$u_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Sol

$$a + 2(0) = -1$$

$$\boxed{a = -1}$$

$$(-1, 0, 1) = -1(1, 1, 0) + 1(0, 1, 1) + 0(2, 2, 3)$$

the matrix representation.

$$[T]_{B_1}^{B_2} = \begin{bmatrix} -1/3 & -1 \\ 0 & 1 \\ 2/3 & 0 \end{bmatrix}$$

31.01.2026.

Saturday.

Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the transformation defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix}$

find the matrix for the transformation T with respect to the matrix basis

$$u_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, u_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

so)

Given $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is linear $B = \{v_1, v_2\}$ is a basis of $\mathbb{R}^2$ $u = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$u_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$B' = \{v_1, v_2, v_3\}$ is the basis of $\mathbb{R}^3$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$T(u_1) = T\left[\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right] = \begin{bmatrix} -5(3) + 13(1) \\ -7(1) + 16(1) \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \end{bmatrix}$$

$$T(u_2) = T\left[\begin{bmatrix} 5 \\ 2 \end{bmatrix}\right] = \begin{bmatrix} -5(5) + 13(2) \\ -7(2) + 16(2) \end{bmatrix} = \begin{bmatrix} -1 \\ 10 \end{bmatrix}$$

$T(u_1)$ & $T(u_2)$ as linear combination of v_1, v_2, v_3 .

$$T(u_1) = \begin{bmatrix} -2 \\ 9 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ 9 \end{bmatrix} = \begin{bmatrix} a - b \\ 2b + c \\ -a + 2b + 2c \end{bmatrix}$$

$$a - b = -2 \rightarrow (1)$$

$$2b + c = 9 \rightarrow (2)$$

$$-a + 2b + 2c = -5 \rightarrow (3)$$

$$a - b = -2$$

$$-b = -2 - a$$

$$\boxed{b = a + 2} \rightarrow (4)$$

$$b = a + 2 \text{ subs in (2)}$$

$$2b + c = 9$$

$$2(a + 2) + c = 9$$

$$2a + 4 + c = 9$$

$$2a + c = 5 \rightarrow (5)$$

$$2a + c = 5$$

$$\boxed{c = 5 - 2a} \rightarrow (6)$$

$$c = 5 - 2a \text{ subs in (3)}$$

$$-a + 2b + 2c = -5$$

$$-a + 2b + 2(5 - 2a) = -5$$

$$-a + 2b - 4a = -5$$

$$2b - 5a = -5 \rightarrow (6)$$

from (1) $\times 2$

$$2a - 2b = 12$$

$$\begin{array}{r} 2b - 5a = -5 \\ (-) \quad (+) \quad (+) \\ \hline \end{array}$$

$$-3a = -3$$

$$a = -3 / -3$$

$$\boxed{a = 1}$$

$a = 1, c = -2$ sub in (3)

$$-a + 2b + 2c = -5$$

$$-1 + 2b + 2(-2) = -5$$

$$2b - 5 = -5$$

$$2b = -5 + 5$$

$$2b = 0$$

$$\boxed{b = 0}$$

$$T(u_1) = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} \Rightarrow 1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$T(u_2) = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} a - b \\ 2b + c \\ -a + 2b + 2c \end{bmatrix}$$

$$a - b = 2 \rightarrow (7)$$

$$2b + c = 1 \rightarrow (8)$$

$$-a + 2b + 2c = -3 \rightarrow (9)$$

$$a - b = 2$$

$$-b = 2 - a$$

$$\boxed{b = a - 2} \rightarrow$$

$b = a - 2$ sub in (8)

$$2(a - 2) + c = 1$$

$$2a - 4 + c = 1$$

$$2a + c = 5$$

$$\boxed{c = 5 - 2a} \rightarrow$$

$c = 5 - 2a$ sub in

$$-a + 2b + 2c = -3$$

$$-a + 2(a - 2) + 2(5 - 2a) = -3$$

$$-a + 2a - 4 + 10 - 4a = -3$$

$$-3a + 6 = -3$$

$$-3a = -3 - 6$$

$$-3a = -9$$

$$/3a = 3$$

$$\boxed{a = 3}$$

$a = 3$ sub in (7)

$$3 - b = 2$$

$$-b = 2 - 3$$

$$-b = -1$$

$$\boxed{b = 1}$$

$a = 3, b = 1$ sub in (9)

$$-a + 2b + 2c = -3$$

$$(-3) + 2(1) + 2c = -3$$

$$-1 + 2c = -3$$

$$2c = -3 + 1$$

$$2c = -2$$

$$c = -2/2$$

$$c = -1$$

$$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$[T]_{B_1}^{B_2} = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 2 & -1 \\ -2 & -2 & 2 \end{bmatrix}$$

02.02.2026

Monday

Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be defined for $T(x, y) = (x+2y, -x, 0)$. Find the matrix of the representation from B' to B with respect to the bases $B = \{u_1, u_2\}$ & $B' = \{v_1, v_2, v_3\}$ where $u_1 = (1, 3)$, $u_2 = (-2, 4)$, $v_1 = (1, 1, 1)$, $v_2 = (2, 2, 0)$, $v_3 = (3, 3, 0)$

sol

$T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is linear transformation.

$$T(x, y) = (x+2y, -x, 0)$$

$$B = \{u_1, u_2\} = (1, 3), (-2, 4)$$

$$B' = \{v_1, v_2, v_3\} = (1, 1, 1), (2, 2, 0), (3, 3, 0)$$

$$T(u_1) = T(1, 3) = (1+2(3), -1, 0)$$

$$= (7, -1, 0)$$

$$T(u_1) = (7, -1, 0)$$

$$T(u_2) = T(-2, 4) = (-2+2(4), 2, 0)$$

$$= (-2+8, 2, 0)$$

$$T(u_2) = (6, 2, 0)$$

$T(u_1) = (7, -1, 0)$ as a linear transformation of $(1, 1, 1)$, $(2, 2, 0)$ & $(3, 3, 0)$

$$(7, -1, 0) = a(1, 1, 1) + b(2, 2, 0) + c(3, 3, 0)$$

$$(7, -1, 0) = (a+2b+3c, a+2b+3c, a+0+0)$$

$$a+2b+3c = 7 \rightarrow (1)$$

$$a+2b+3c = -1 \rightarrow (2)$$

$$a = 0 \rightarrow (3)$$

$$2 \times 2$$

$$a+2b+3c = 7$$

$$a+2b+3c = -1$$

Both equations have no solutions since they are dependent. We cannot write the matrix representation of the linear transformation.

Find the linear transformation $T: V_3(\mathbb{R}) \rightarrow V_3(\mathbb{R})$ determined by the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \text{ with respect to the standard basis } \{e_1, e_2, e_3\}.$$

sol

$T: V_3(\mathbb{R}) \rightarrow V_3(\mathbb{R})$ is linear.

$$T = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

$T: V_3(\mathbb{R}) \rightarrow V_3(\mathbb{R})$

$$T(e_1) = (1, 2, 1)$$

$$T(e_2) = (0, 1, 1)$$

$$T(e_3) = (-1, 3, 4)$$

$$a, b, c \in \mathbb{R}$$

$$a, b, c = ae_1 + be_2 + ce_3$$

$$\begin{aligned} T(a, b, c) &= a(Te_1) + b(Te_2) + c(Te_3) \\ &= a(1, 2, 1) + b(0, 1, 1) + c(-1, 3, 4) \end{aligned}$$

$$T(a, b, c) = (a - c, 2a + b + 3c, a + b + 4c)$$

Thus the required linear transformation.

$T: V_3(\mathbb{R}) \rightarrow V_3(\mathbb{R})$ is

$$T(a, b, c) = (a - c, 2a + b + 3c, a + b + 4c)$$

Find the linear transformation $T: V_2(\mathbb{R}) \rightarrow V_2(\mathbb{R})$ determined by the matrix

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \text{ with respect to the standard basis } \{e_1, e_2\}.$$

sol

$T: V_2(\mathbb{R}) \rightarrow V_2(\mathbb{R})$

$$T = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$T(e_1) = (2, 1, -1)$$

$$T(e_2) = (1, 1, 1)$$

$$a, b \in \mathbb{R}$$

$$a, b = ae_1 + be_2$$

$$\begin{aligned} T(a, b) &= a(2, 1, -1) + b(1, 1, 1) \\ &= (2a + b, a + b, -a + b) \end{aligned}$$

$$T(a, b) = (2a + b, a + b, -a + b)$$

Thus the required linear transformation

$T: V_2(\mathbb{R}) \rightarrow V_2(\mathbb{R})$ is

$$T(a, b) = (2a + b, a + b, -a + b)$$

obtain the linear transformation determined the matrix $T: V_2(\mathbb{R}) \rightarrow V_2(\mathbb{R})$ given by

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \text{ with respect to standard basis.}$$

So,

$$T: V_2(\mathbb{R}) \rightarrow V_2(\mathbb{R})$$

$$T = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$T(e_1) = (\cos \theta, -\sin \theta)$$

$$T(e_2) = (\sin \theta, \cos \theta)$$

$$a, b \in V_2(\mathbb{R})$$

$$a, b = a(e_1) + b(e_2)$$

$$T(a, b) = a(Te_1) + b(Te_2)$$

$$= a(\cos \theta, -\sin \theta) + b(\sin \theta, \cos \theta)$$

$$T(a, b) = a \cos \theta + b \sin \theta, -a \sin \theta + b \cos \theta$$

Thus the required linear transformation.

$$T: V_2(\mathbb{R}) \rightarrow V_2(\mathbb{R}) \text{ is}$$

$$T(a, b) = a \cos \theta + b \sin \theta, -a \sin \theta + b \cos \theta$$

28.02.2026
Tuesday.

Cauchy - Schwarz inequality & Triangle inequality.

Theorem:

Let V be an inner product space over F then for all $x, y \in V$ & $c \in F$ the following statements are true.

$$a) \|cx\| = |c| \|x\|$$

$$b) \|x\| = 0 \text{ iff } x = 0$$

$$c) |\langle x, y \rangle| \leq \|x\| \|y\| \quad (C.S.I.)$$

$$d) \|x+y\| \leq \|x\| + \|y\| \quad (T.I.)$$

Proof.

$$\text{By the definition } \|x\| = \sqrt{\langle x, x \rangle}$$

$$a) \text{ to prove } \|cx\| = |c| \|x\|$$

$$\|cx\| = \sqrt{\langle cx, cx \rangle}$$

$$\|cx\|^2 = \langle cx, cx \rangle$$

$$= c \bar{c} \langle x, x \rangle$$

$$= |c|^2 \|x\|^2$$

$$\|cx\|^2 = |c|^2 \|x\|^2$$

$$\|cx\| = |c| \|x\|$$

$$b) \text{ to prove } \|x\| = 0 \text{ iff } x = 0$$

$$\|x\| = 0$$

$$\sqrt{\langle x, x \rangle} = 0$$

$$\langle x, x \rangle = 0$$

$$x = 0$$

$$\text{since } \langle x, x \rangle = 0 \text{ iff } x = 0$$

Eigenvalues and Eigenvectors

Definition:

Let A be a $n \times n$ square matrix. A number λ is called an eigenvalue of A if there exists a non-zero vector $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ such that $Ax = \lambda x$ and x is called an eigenvector corresponding to the eigenvalue of A .

$$|A - \lambda I| = 0$$

① Find the sum and product of the eigenvalues of

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Sol

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Sum of the eigenvalues of $A = \text{trace}(A)$

= sum of the elements of leading diagonal

$$= 8 + 7 + 3 \\ = 18$$

Product of the eigenvalues of A is

$$|A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$= 8(21 - 16) + 6(-18 + 8) + 2(24 - 14)$$

$$= 8(5) + 6(-10) + 2(10)$$

$$= 40 - 60 + 20$$

$$|A| = 0$$

Sum of the eigenvalues of $A = 18$

Product of the eigenvalues of $A = 0$

② The product of two eigenvalues of the matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \text{ is } 16. \text{ Find the third eigenvalue and sum of the eigenvalues.}$$

Sol

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of A

Product of two eigenvalues = 16

$$\lambda_1 \lambda_2 = 16$$

Product of the eigenvalues of A is

$$\lambda_1 \lambda_2 \lambda_3 = |A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 6(9-1) + 2(-6+2) + 2(9-6) \\ = 6(8) + 2(-4) + 2(3) \\ = 48 - 8 + 6 \\ = 46$$

$$\lambda_1 \lambda_2 \lambda_3 = 16 \cdot 1$$

$$\lambda_3 = \frac{|A|}{\lambda_1 \lambda_2} \Rightarrow \lambda_3 = \frac{46}{16} = 2.875$$

Third eigenvalue is 2.875.

Sum of the eigenvalues = trace(A)

$$= 6 + 3 + 3 = 12$$

③ If 0, 3, 15 are eigenvalues of A =

Find the eigenvalues of A^2 .

Sol

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$(\lambda_1, \lambda_2, \lambda_3) = (0, 3, 15)$ are eigenvalues of A

$$A = (0, 3, 15)$$

Eigenvalues of $A^2 = (0, 9, 225)$

Q. If the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \text{ are } 2, 2, 8. \text{ Find the eigenvalues of}$$

$$A^{-1}, A^3, A^t$$

Sol

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Eigenvalues of A are $\lambda_1 = 2, \lambda_2 = 2, \lambda_3 = 8$

" " $A^{-1} = \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$

$$A^{-1} = \frac{1}{2}, \frac{1}{2}, \frac{1}{8}$$

$$A^3 = (2)^3, (2)^3, (8)^3$$

Eigenvalues of A^3 $A^3 = 8, 8, 512$

Eigenvalues of $A^t =$ Eigenvalues of A

" " $= 2, 2, 8 //$

Q. If $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$, find the eigenvalues of A^2 and A^{-1}

Sol

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

Eigenvalues of A are $\lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 5$

" " $A^2 = 9, 4, 25$

" " $A^{-1} = \frac{1}{3}, \frac{1}{2}, \frac{1}{5}$

Algebraic multiplicity:

The algebraic multiplicity of an eigenvalue is the number of times it appears as a root of the characteristic polynomial.

Invertible Matrix:

We define invertible matrices as square matrices whose inverse exists. They are non-singular matrices as their determinant exists.

$$AB = BA = I$$

①. Find the multiplicity of all eigenvalues of

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

Sol

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

Eigenvalues

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)^2 (4-\lambda) = 0$$

$$\lambda = 3, \lambda = 4$$

$\therefore$ algebraic multiplicity

$$\lambda = 3 \text{ is } 2$$

Algebraic multiplicity

$$\lambda = 4 \text{ is } 1$$

$$(3-\lambda)((3-\lambda)(4-\lambda)) - 1(0-0) + 0(0) = 0$$

$$(3-\lambda)(12-3\lambda+4\lambda+\lambda^2) = 0$$

$$(3-\lambda)(12-7\lambda+\lambda^2) = 0$$

$$36 - 21\lambda + 3\lambda^2 - 12\lambda + 7\lambda - \lambda^3 = 0$$

$$-\lambda^3 + 3\lambda^2 - 26\lambda + 36 = 0$$

$$(3-\lambda)((3-\lambda)(4-\lambda))$$

$$-\frac{2}{3}$$

$$-\frac{7}{6}$$

$$x \Rightarrow -\lambda^3 - 3\lambda^2 + 26\lambda + 26 = 0$$

$$\begin{array}{c|ccc} 2 & 1 & -3 & 26 & 26 \\ & 0 & 2 & & \end{array}$$

Φ

⑦ Let $A, B \in M_{n \times n}(\mathbb{C})$. Find 2×2 non-zero matrices A and B such that both A and $A + cB$ are invertible for all $c \in \mathbb{C}$.

Sol

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1 \neq 0 \quad A \text{ is invertible.}$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$A + cB$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ c & 1+c \end{bmatrix}$$

$$A + cB = 1 + c - c$$

$$= 1$$

$A + cB \neq 0$ $A + cB$ is invertible.

⑧ Check $A = \begin{bmatrix} 5 & 6 \\ 4 & 5 \end{bmatrix}$ is invertible matrix if $B = \begin{bmatrix} 5 & -6 \\ -4 & 5 \end{bmatrix}$

is the inverse.

Sol

$$A = \begin{bmatrix} 5 & 6 \\ 3 & 5 \end{bmatrix}, B = \begin{bmatrix} 5 & -6 \\ -4 & 5 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 & 6 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 5 & -6 \\ -4 & 5 \end{bmatrix} = \begin{bmatrix} 25 - 24 & -30 + 30 \\ 15 - 20 & -18 + 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -5 & 7 \end{bmatrix}$$

$$BA = \begin{bmatrix} 5 & -6 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 25 - 18 & 30 - 30 \\ -20 + 15 & -24 + 25 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ -5 & 1 \end{bmatrix}$$

④ Find the values of a for which the matrix

$$\begin{bmatrix} 1 & a & 0 \\ a & 1 & a \\ 0 & a & 1 \end{bmatrix}$$

Sol

$$\begin{bmatrix} 1 & a & 0 \\ a & 1 & a \\ 0 & a & 1 \end{bmatrix} \neq 0$$

$$1(1-a^2) - a(a-0) + 0 \neq 0$$

$$1-a^2-a^2 \neq 0$$

$$1-2a^2 \neq 0$$

$$2a^2 \neq 1$$

$$a^2 \neq \frac{1}{2}$$

$$a \neq \pm \frac{1}{\sqrt{2}}$$

1st Row:-

$$\text{co-factor of } 1 = \begin{vmatrix} 1 & a \\ a & 1 \end{vmatrix} = 1-a^2$$

$$\text{co-factor of } a = -\begin{vmatrix} a & a \\ 0 & 1 \end{vmatrix} = -a$$

$$\text{co-factor of } 0 = \begin{vmatrix} a & 1 \\ 0 & a \end{vmatrix} = a^2$$

2nd Row

$$\text{co-factor of } a = -\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} = -a$$

$$\text{co-factor of } 1 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\text{co-factor of } a = -\begin{vmatrix} 1 & a \\ 0 & a \end{vmatrix} = -a$$

3rd Row

$$\text{co-factor of } 0 = \begin{vmatrix} a & 0 \\ 1 & a \end{vmatrix} = a^2$$

$$\text{co-factor of } a = -\begin{vmatrix} 1 & 0 \\ a & a \end{vmatrix} = -a$$

$$\text{co-factor of } 1 = \begin{vmatrix} 1 & a \\ a & 1 \end{vmatrix} = 1-a^2$$

$$\text{co-factor of } A = \begin{bmatrix} 1-a^2 & -a & a^2 \\ -a & 1 & -a \\ a^2 & -a & 1-a^2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$= \frac{1}{1-2a^2} \begin{bmatrix} 1-a^2 & -a & a^2 \\ -a & 1 & -a \\ a^2 & -a & 1-a^2 \end{bmatrix}$$

Q.12. Find the sum & product of the Eigenvalues of the Matrix $A = \begin{bmatrix} 2 & -3 \\ A & -2 \end{bmatrix}$; Ans: Sum = 0, product = 8

Q.2. Find the sum & product of all the Eigenvalues of the Matrix $\begin{bmatrix} 1 & 2 & -2 \\ 1 & 0 & 3 \\ -2 & -1 & -3 \end{bmatrix}$; Ans: Sum = -2, product = -1

Q.3. Find the Eigen values & eigenvectors for the Matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = 0$$

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 0 \\ 2 & 2 & 3-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)((2-\lambda)(3-\lambda)) - 0(0) - 1(2 - (2-\lambda)) = 0$$

$$(1-\lambda)((2-\lambda)(3-\lambda)) - 1(2 - 2 + \lambda) = 0$$

$$(1-\lambda)((2-\lambda)(3-\lambda)) - 1(-2 + 2\lambda) = 0$$

$$(1-\lambda)(6 - 2\lambda - 3\lambda + \lambda^2) + 2 - 2\lambda = 0$$

$$6 - 2\lambda - 3\lambda + \lambda^2 - 6\lambda + 2\lambda^2 + 3\lambda^2 - \lambda^3 = 0$$

$$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\lambda = 1, \lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 3)(\lambda - 2) = 0$$

$$\lambda = 1, \lambda = 3, \lambda = 2$$

Eigen Values

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ & & -5 & 6 & 6 \end{array}$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$\begin{array}{c} 6 \\ \wedge \\ -3-2 \\ \vee \\ -5 \end{array}$$

Eigenwert

$$|A - \lambda I| = 0$$

$$\lambda = 1$$

$$\begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{vmatrix} = 0$$

$$0(0-0) - 0(0) - 1(2-0) = 0$$

Eigenvektoren:

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case (P) $\lambda = 1$

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x_1 + 0x_2 - x_3 = 0 \Rightarrow 0$$

$$x_1 + x_2 + x_3 = 0 \Rightarrow 0$$

$$2x_1 + 2x_2 + 2x_3 = 0 \Rightarrow 0$$

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{array}$$

$$\frac{x_1}{0+1} = \frac{x_2}{-1-0} = \frac{x_3}{0-0}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0}$$

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 0$$

$$-1(0-2) - 0(0) - 1(2-0) = 0$$

Case (P) $\lambda = 2$

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + 0x_2 - x_3 = 0$$

$$x_1 + 0x_2 + x_3 = 0$$

$$2x_1 + 2x_2 + x_3 = 0$$

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}$$

$$\frac{x_1}{0-0} = \frac{x_2}{-1+1} = \frac{x_3}{0-0}$$

$$\frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$x_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case (P) $\lambda = 3$

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + 0x_2 - x_3 = 0$$

$$x_1 - x_2 + x_3 = 0$$

$$2x_1 + 2x_2 + 0x_3 = 0$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ 0 & -1 & -2 & 0 \\ -1 & 1 & 1 & 2 \\ \frac{x_1}{0-1} = \frac{x_2}{-1+2} = \frac{x_3}{-1-0} \end{array}$$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$x = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

Eigen vectors

$$x = [x_1, x_2, x_3]$$

$$x = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Find the eigenvalues and eigen vectors of the matrix

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -1 & 2 & -2 \end{bmatrix}$$

Sol

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -1 & 2 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = 0$$

$$\begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ -1 & 2 & -3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)((1-\lambda)(-3-\lambda)-2)-2$$

$$(2(-3-\lambda)+1)=0$$

$$(2-\lambda)[(-3-\lambda+3\lambda+\lambda^2)-2]$$

$$-2[-6-2\lambda+1]=0$$

$$(2-\lambda)[-3-\lambda+3\lambda+\lambda^2-2]$$

$$-2[-6-2\lambda+1]=0$$

$$(2-\lambda)[- \lambda+3\lambda+\lambda^2-5]$$

$$+12+4\lambda-14=0$$

$$-2\lambda+6\lambda+2\lambda^2-10+\lambda^2-3\lambda^2$$

$$-\lambda^3+5\lambda+12+4\lambda-14=0$$

$$-\lambda^3+0\lambda^2+13\lambda-12=0 \quad -\frac{24}{12}$$

$$x \Rightarrow \lambda^3-13\lambda+12=0$$

$$\begin{array}{cccc} 1 & 14 & 0 & -13 & 12 \\ & 0 & 1 & 1 & -12 \\ & & 1 & 1 & -12 \\ & & & & 0 \end{array}$$

$$\lambda^2+\lambda-12=0 \quad -12$$

$$(\lambda+4)(\lambda-3)=0$$

$$\lambda+4=0 \quad \lambda-3=0$$

$$\lambda=-4 \quad \lambda=3, \lambda=1$$

Eigen vectors:

$$(A-\lambda I)x=0$$

$$\begin{bmatrix} 2-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ -1 & 2 & -3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Case (P)} \quad \lambda = -4$$

$$\begin{bmatrix} 6 & 2 & 0 \\ 2 & 5 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$6x_1 + 2x_2 + 0x_3 = 0$$

$$2x_1 + 5x_2 + x_3 = 0$$

$$-1x_1 + 2x_2 + x_3 = 0$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ 2 & 0 & 6 \\ 5 & 1 & 2 \end{array}$$

$$\frac{x_1}{2-0} = \frac{x_2}{0-6} = \frac{x_3}{30-4}$$

$$\frac{x_1}{2} = \frac{x_2}{-6} = \frac{x_3}{26}$$

$$\therefore \frac{x_1}{1} = \frac{x_2}{-3} = \frac{x_3}{13}$$

$$x_1 = \begin{bmatrix} 2 \\ -3 \\ 13 \end{bmatrix}$$

Case (ii) Sub $\lambda = 1$ in (B)

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ -1 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 + 0x_3 = 0$$

$$2x_1 + 0x_2 + x_3 = 0$$

$$-x_1 + 2x_2 - 4x_3 = 0$$

$$x_1 \quad x_2 \quad x_3$$

$$2 \quad 0 \quad 1 \quad 2$$

$$0 \quad 1 \quad 2 \quad 0$$

$$\frac{x_1}{2-0} = \frac{x_2}{0-1} = \frac{x_3}{0-4}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{-4}$$

$$x_2 = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix}$$

Case (iii)

Sub $\lambda = 3$ in (B)

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & -2 & 1 \\ -1 & 2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + 2x_2 + 0x_3 = 0$$

$$+2x_1 - 2x_2 + x_3 = 0$$

$$-x_1 + 2x_2 - 6x_3 = 0$$

$$x_1 \quad x_2 \quad x_3$$

$$2 \quad 0 \quad -1 \quad 2$$

$$-2 \quad 1 \quad 2 \quad -2$$

$$\frac{x_1}{2-0} = \frac{x_2}{0+1} = \frac{x_3}{2-4}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$$x_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Eigen vectors

$$x = [x_1, x_2, x_3]$$

$$x = \begin{bmatrix} 1 & 2 & 2 \\ -3 & -1 & 1 \\ 13 & -4 & -2 \end{bmatrix}$$

HW

① Find the eigenvalues & eigen vectors of the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Sol: Ans:

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\lambda = 1 \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\lambda = 2 \Rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda = 3 \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

② Let be a linear transformation from R^3 whose matrix rep A w.r to the standard basis is

$$\begin{bmatrix} 1 & 3 & -3 \\ 3 & 1 & -3 \\ -3 & -3 & 1 \end{bmatrix}$$

Find Eigen values & vectors.

Sol
 $\lambda = -2, +2, +7$

$\lambda = -2 \Rightarrow \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\lambda = 2 \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 7 \Rightarrow \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

How
 Find the rank and nullity
 of the matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 9 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 1 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

Ans:

$\alpha_3 (4, 2, 1, 0, 0, 0) +$

$\alpha_4 (28, 12, 0, 1, 0, 0) +$

$\alpha_5 (37, 16, 0, 0, 1, 0) +$

$\alpha_6 (-13, -5, 0, 0, 0, 1)$

Find the basis for the
 nullspace of

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Ans: $x_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

17/2/26:

MATHS
Unit - A

Ques

Matrix Decomposition

- 1) Write the matrix of the quadratic form $2x_1^2 - 2x_2^2 + 4x_3^2 + 2x_1x_2 - 6x_1x_3 + 6x_2x_3$

Sol

$$a = \begin{bmatrix} \text{coeff } x_1^2 & \frac{1}{2} \text{coeff } x_1x_2 & \frac{1}{2} \text{coeff } x_1x_3 \\ \frac{1}{2} \text{coeff } x_2x_1 & \text{coeff } x_2^2 & \frac{1}{2} \text{coeff } x_2x_3 \\ \frac{1}{2} \text{coeff } x_3x_1 & \frac{1}{2} \text{coeff } x_3x_2 & \text{coeff } x_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & \frac{1}{2}(2) & \frac{1}{2}(-6) \\ \frac{1}{2}(2) & -2 & \frac{1}{2}(4) \\ \frac{1}{2}(-6) & \frac{1}{2}(4) & 4 \end{bmatrix}$$

$$Q = \begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 2 \\ -3 & 2 & 4 \end{bmatrix}$$

- 2) Write the matrix of the quadratic form $2x^2 + 8z^2 + 4xy + 6xz - 2yz$.

Sol

$$a = \begin{bmatrix} \text{coeff } x^2 & \frac{1}{2} \text{coeff } xy & \frac{1}{2} \text{coeff } xz \\ \frac{1}{2} \text{coeff } yx & \text{coeff } y^2 & \frac{1}{2} \text{coeff } yz \\ \frac{1}{2} \text{coeff } zx & \frac{1}{2} \text{coeff } zy & \text{coeff } z^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & \frac{1}{2}(4) & \frac{1}{2}(6) \\ \frac{1}{2}(4) & 0 & \frac{1}{2}(-2) \\ \frac{1}{2}(6) & \frac{1}{2}(-2) & 8 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 3 \\ 2 & 0 & -1 \\ 3 & -1 & 8 \end{bmatrix}$$

3) Find the nature of the matrix of the quadratic form
 $11x^2 + 2y^2 + 2z^2 + 4xy - 2yz + 4xz$.

Sol

$$a = \begin{bmatrix} \text{Coeff } x^2 & \frac{1}{2} \text{Coeff } xy & \frac{1}{2} \text{Coeff } xz \\ \frac{1}{2} \text{Coeff } yx & \text{Coeff } y^2 & \frac{1}{2} \text{Coeff } yz \\ \frac{1}{2} \text{Coeff } zx & \frac{1}{2} \text{Coeff } zy & \text{Coeff } z^2 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & \frac{1}{2}(4) & \frac{1}{2}(4) \\ \frac{1}{2}(4) & 2 & \frac{1}{2}(-2) \\ \frac{1}{2}(4) & \frac{1}{2}(-2) & 2 \end{bmatrix}$$

$$a = \begin{bmatrix} 11 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}$$

$$P_1 = a_{11} = [11] = +ve$$

$$P_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 11 & 2 \\ 2 & 2 \end{bmatrix} = (2 \cdot 2 - 4) = 0 = +ve.$$

$$P_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 11 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}$$

$$= 11(4-1) - 2(4+2) + 2(-2-4)$$

$$= 11(3) - 2(6) + 2(-6)$$

$$= 33 - 12 - 12$$

$$= 9 \Rightarrow +ve$$

The matrix is positive definite.

4) Find the nature of the matrix of the quadratic form.
 $6x^2 + 2y^2 + 3z^2 - 4xy + 8xz.$

Sol

$$A = \begin{bmatrix} \text{coeff } x^2 & \frac{1}{2} \text{coeff } xy & \frac{1}{2} \text{coeff } xz \\ \frac{1}{2} \text{coeff } yx & \text{coeff } y^2 & \frac{1}{2} \text{coeff } yz \\ \frac{1}{2} \text{coeff } zx & \frac{1}{2} \text{coeff } zy & \text{coeff } z^2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & \frac{1}{2}(-4) & \frac{1}{2}(8) \\ \frac{1}{2}(-4) & 2 & \frac{1}{2}(0) \\ \frac{1}{2}(8) & \frac{1}{2}(0) & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 & 4 \\ -2 & 2 & 0 \\ 4 & 0 & 3 \end{bmatrix}$$

$$P_1 = a_{11} = [6] = +ve.$$

$$P_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix} = (12 - 4) = 8 = +ve.$$

$$P_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 6 & -2 & 4 \\ -2 & 2 & 0 \\ 4 & 0 & 3 \end{bmatrix}$$

$$= 6(6-0) + 2(-6-0) + 4(0-8)$$

$$= 6(6) + 2(-6) + 4(-8)$$

$$= 36 - 12 - 32$$

$$= -8 \Rightarrow -ve.$$

The matrix is indefinite.

- 4) Determine whether the given matrix is positive definite or not.

$$A = \begin{bmatrix} 2 & 10 & -2 \\ 10 & 5 & 8 \\ -2 & 8 & 11 \end{bmatrix}$$

Sol

$$P_1 = a_{11} = [2] = +ve$$

$$P_2 = \begin{bmatrix} 2 & 10 \\ 10 & 5 \end{bmatrix} = 10 - 100 = -90 = -ve$$

$$\begin{aligned} P_3 &= \begin{bmatrix} 2 & 10 & -2 \\ 10 & 5 & 8 \\ -2 & 8 & 11 \end{bmatrix} = 2(55 - 64) - 10(110 + 16) + (-2)(80 + 10) \\ &= 2(-9) - 10(126) - 2(90) \\ &= -18 - 1260 - 180 \\ &= -1458 = -ve. \end{aligned}$$

The matrix is indefinite and not positively definite.

- 5) Find the nature of the matrix of the quadratic form

$$x_1^2 + 2x_2^2$$

Sol

Quadratic form is $x_1^2 + 2x_2^2$

$$a = \begin{bmatrix} \text{coeff } x_1^2 & \frac{1}{2} \text{coeff } x_1 x_2 & \frac{1}{2} \text{coeff } x_1 x_3 \\ \frac{1}{2} \text{coeff } x_2 x_1 & \text{coeff } x_2^2 & \frac{1}{2} \text{coeff } x_2 x_3 \\ \frac{1}{2} \text{coeff } x_3 x_1 & \frac{1}{2} \text{coeff } x_3 x_2 & \text{coeff } x_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P_1 = 1 = +ve$$

$$P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = 2 - 0 = 2 = +ve$$

$$P_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 = +ve$$

The matrix is semi positively definite.

D) The Quadratic form $2x^2 + 2xz + 3z^2$.

sol

$$Q = \begin{bmatrix} \text{Coeff } x^2 & \frac{1}{2} \text{Coeff } xy & \frac{1}{2} \text{Coeff } xz \\ \frac{1}{2} \text{Coeff } yx & \text{Coeff } y^2 & \frac{1}{2} \text{Coeff } yz \\ \frac{1}{2} \text{Coeff } zx & \frac{1}{2} \text{Coeff } zy & \text{Coeff } z^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & \frac{1}{2}(0) & \frac{1}{2}(2) \\ \frac{1}{2}(0) & 0 & \frac{1}{2}(0) \\ \frac{1}{2}(2) & \frac{1}{2}(0) & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

$$P_1 = a_{11} = 2 \Rightarrow +ve$$

$$P_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = (2 - 0) = 0 \Rightarrow +ve$$

$$P_3 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 3 \end{bmatrix} = 2(0 \cdot 0) - (0) + 1(0) = 0 \Rightarrow +ve$$

2) Find the nature of Matrix of quadratic form
 $-2x_1^2 + 5x_2^2 + 3x_3^2 + 4x_1x_2$

Sol

$$A = \begin{bmatrix} \text{Coeff } x_1^2 & \frac{1}{2} \text{Coeff } x_1x_2 & \frac{1}{2} \text{Coeff } x_1x_3 \\ \frac{1}{2} \text{Coeff } x_2x_1 & \text{Coeff } x_2^2 & \frac{1}{2} \text{Coeff } x_2x_3 \\ \frac{1}{2} \text{Coeff } x_3x_1 & \frac{1}{2} \text{Coeff } x_3x_2 & \text{Coeff } x_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & \frac{1}{2}(4) & \frac{1}{2}(0) \\ \frac{1}{2}(4) & 5 & \frac{1}{2}(0) \\ \frac{1}{2}(0) & \frac{1}{2}(0) & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$P_1 = a_{11} = -2 \Rightarrow +ve$$

$$P_2 = \begin{vmatrix} -2 & 2 \\ 2 & 5 \end{vmatrix} = (10 - 4) = 6 \Rightarrow +ve$$

$$\begin{aligned} P_3 &= \begin{vmatrix} -2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{vmatrix} = -2(15 - 0) - 2(6 - 0) + 0(0 - 0) \\ &= -2(15) - 2(6) + 0 \\ &= -30 - 12 \\ &= -42 \Rightarrow +ve. \end{aligned}$$

All are positive

3) Find the nature of the matrix.

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Sol

$$P_1 = a_{11} = 6 \Rightarrow +ve$$

$$P_2 = \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix} = (18 - 4) = 14 \Rightarrow +ve$$

$$\begin{aligned} P_3 &= \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = 6(9-1) + 2(-6+2) + 2(2-6) \\ &= 6(8) + 2(-4) + 2(-4) \\ &= 48 - 8 - 8 \\ &= 48 - 16 \end{aligned}$$

$$P_3 = 32 \Rightarrow +ve.$$

All are positive.

4) Find the nature of the matrix $\begin{bmatrix} -3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & -3 \end{bmatrix}$

Sol

$$P_1 = a_{11} = -3 = -ve$$

$$P_2 = \begin{bmatrix} -3 & -1 \\ -1 & 5 \end{bmatrix} = (-15 - 1) = -16 \Rightarrow -ve$$

$$P_3 = \begin{bmatrix} -3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & -3 \end{bmatrix}$$

$$\begin{aligned}
 &= -3(-15-1) + 1(3+1) + 1(1-5) \\
 &= -3(-16) + 1(4) + 1(-4) \\
 &= +48 + 4 - 4
 \end{aligned}$$

$$P_3 = 48$$

The matrix is indefinite.

18/2/26.

1. orthogonal transformation of a symmetric matrix to diagonal form and Reduction of quadratic form to canonical form. characteristic equation

$$2 \times 2 \Rightarrow \lambda^2 - S_1 \lambda + S_2 = 0$$

$$3 \times 3 \Rightarrow \lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0$$

Sum

D) Reduces the quadratic form $Q = 6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4zx$ into Canonical form by an orthogonal transformation.

Sol

$$a = \begin{bmatrix} \text{Coeff } x^2 & 1/2 \text{ Coeff } xy & 1/2 \text{ Coeff } xz \\ 1/2 \text{ Coeff } yx & \text{Coeff } y^2 & 1/2 \text{ Coeff } yz \\ 1/2 \text{ Coeff } zx & 1/2 \text{ Coeff } zy & \text{Coeff } z^2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 1/2(-4) & 1/2(4) \\ 1/2(-4) & 3 & -1/2(-2) \\ 1/2(4) & 1/2(-2) & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

characteristics equation

$$|A - \lambda I| = 0$$

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0.$$

S_1 = Sum of the main diagonal elements.

$$S_1 = 6 + 3 + 3 = 12$$

$$S_1 = 12$$

S_2 = Sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$$

$$= (9-1) + (18-4) + (18-4)$$

$$= 8 + 14 + 14$$

$$S_2 = 36$$

$$S_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 6(9-1) + 2(-6+2) + 2(2-6)$$
$$= 6(8) + 2(-4) + 2(-4)$$

$$= 48 - 8 - 8$$

$$S_3 = 32$$

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0$$

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$$\begin{array}{l|cccc} 2 & 1 & -12 & 36 & -32 \\ & 0 & 2 & -20 & 32 \\ \hline 2 & 1 & -10 & 16 & 0 \\ & 0 & 2 & -16 & \\ \hline & 1 & -8 & 0 & \end{array}$$

$$\lambda = 2, \lambda = 2, \lambda - 8 = 0$$

$$\lambda = 2, 2, 8$$

$$|A - \lambda I| = 0$$

$$\lambda = 8$$

$$\left| \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$= \begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 - 2x_2 + 2x_3 = 0 \rightarrow \textcircled{1}$$

$$-2x_1 - 5x_2 - x_3 = 0 \rightarrow \textcircled{2}$$

$$2x_1 - x_2 - 5x_3 = 0 \rightarrow \textcircled{3}$$

$$\frac{x_1}{-2 \ 2} = \frac{x_2}{+2 \ -2} = \frac{x_3}{-2 \ -2}$$

$$\frac{x_1}{2+10} = \frac{x_2}{-4+2} = \frac{x_3}{10-4}$$

$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6}$$

$\div \text{by } 6$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$x_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\lambda = 2$$

$$\begin{bmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x_1 - 2x_2 + 2x_3 = 0 \rightarrow \textcircled{1}$$

$$-2x_1 + x_2 - x_3 = 0 \rightarrow \textcircled{2}$$

$$2x_1 - x_2 + x_3 = 0 \rightarrow \textcircled{3}$$

$$\frac{x_1}{-2} = \frac{x_2}{2} = \frac{x_3}{4}$$

$$\frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{4}$$

$$\frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{0}$$

Let

$$x_1 = 1, x_2 = 1 \text{ sub in } \textcircled{1}$$

$$4x_1 - 2x_2 + 2x_3 = 0$$

$$4 - 2 + 2x_3 = 0$$

$$2 + 2x_3 = 0$$

$$2x_3 = -2$$

$$\boxed{x_3 = -1}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$2l - m + n = 0 \rightarrow (1)$$

$$l + m - n = 0 \rightarrow (2)$$

$$\frac{l}{-1 \ 1} = \frac{m}{1 \ 2} = \frac{n}{2 \ -1}$$

$$1 \ -1 \quad -1 \ 1 \quad 1 \ 1$$

$$\frac{l}{1-1} = \frac{m}{1+2} = \frac{n}{2+1}$$

$$\frac{l}{0} = \frac{m}{3} = \frac{n}{3}$$

$$x_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

Normalized Matrix.

$$\begin{aligned} \|x_1\|^2 &= (2, -1, 1) (2, -1, 1) \\ &= 4 + 1 + 1 \\ &= 6. \end{aligned}$$

$$\|x_1\| = \sqrt{6}.$$

$$\begin{aligned} \|x_2\|^2 &= (1, 1, -1) (1, 1, -1) \\ &= 1 + 1 + 1 \\ &= 3 \end{aligned}$$

$$\|x_2\| = \sqrt{3}$$

$$\begin{aligned} \|x_3\|^2 &= (0, 3, 3) (0, 3, 3) \\ &= 0 + 9 + 9 \\ &= 18 \end{aligned}$$

$$\|x_3\| = \sqrt{18}$$

$$P = \left[\frac{x_1}{\|x_1\|} \quad \frac{x_2}{\|x_2\|} \quad \frac{x_3}{\|x_3\|} \right]$$

$$P = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{3}{\sqrt{18}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{3}{\sqrt{18}} \end{bmatrix}$$

$$P^T = \begin{bmatrix} \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{3}{\sqrt{18}} & \frac{3}{\sqrt{18}} \end{bmatrix}$$

$$AP^T = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{6}} & +\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{3}{\sqrt{18}} \\ \frac{1}{\sqrt{6}} & \frac{3}{\sqrt{18}} & \frac{3}{\sqrt{18}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{12}{\sqrt{6}} + \frac{2}{\sqrt{6}} + \frac{2}{\sqrt{6}} & \frac{6}{\sqrt{3}} - \frac{2}{\sqrt{3}} - \frac{2}{\sqrt{3}} & 0 - \frac{6}{\sqrt{18}} + \frac{6}{\sqrt{18}} \\ \frac{-4}{\sqrt{6}} - \frac{3}{\sqrt{6}} - \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} + \frac{3}{\sqrt{3}} + \frac{1}{\sqrt{3}} & 0 + \frac{9}{\sqrt{18}} - \frac{3}{\sqrt{18}} \\ \frac{4}{\sqrt{6}} + \frac{1}{\sqrt{6}} + \frac{3}{\sqrt{6}} & \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}} - \frac{3}{\sqrt{3}} & 0 - \frac{3}{\sqrt{18}} + \frac{9}{\sqrt{18}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{16}{\sqrt{6}} & \frac{2}{\sqrt{3}} & 0 \\ \frac{-8}{\sqrt{6}} & \frac{2}{\sqrt{3}} & \frac{6}{\sqrt{18}} \\ \frac{8}{\sqrt{6}} & -\frac{2}{\sqrt{3}} & \frac{6}{\sqrt{18}} \end{bmatrix}$$

$$D = AP P^T$$

$$AP = \begin{bmatrix} \frac{16}{\sqrt{18}} & \frac{2}{\sqrt{3}} & 0 \\ -\frac{8}{\sqrt{6}} & \frac{2}{\sqrt{3}} & \frac{6}{\sqrt{18}} \\ \frac{8}{\sqrt{6}} & \frac{2}{\sqrt{3}} & \frac{6}{\sqrt{18}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{18}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{3}{\sqrt{18}} & \frac{3}{\sqrt{18}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{32+8+8}{6} & \frac{4-2-2}{\sqrt{18}} & \frac{0-6+6}{\sqrt{108}} \\ \frac{16-8-8}{\sqrt{18}} & \frac{2+2+2}{3} & \frac{0+6-6}{\sqrt{54}} \\ \frac{0-24-24}{\sqrt{108}} & \frac{0+6-6}{\sqrt{54}} & \frac{0+18+18}{18} \end{bmatrix}$$

$$D = \begin{bmatrix} \frac{48}{6} & 0 & 0 \\ 0 & \frac{6}{3} & 0 \\ 0 & 0 & \frac{36}{18} \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Canonical form

$$(y_1, y_2, y_3) \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 8y_1^2 + 2y_2^2 + 2y_3^2$$

Nature : positive definite

Rank : 3

Index : 3

Signature : 3

2. Reduce the quadratic form is $3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$ to canonical form. Discuss its nature.

Sol

$$A = \begin{bmatrix} 3 & \frac{1}{2}(2) & \frac{1}{2}(2) \\ \frac{1}{2}(2) & 3 & \frac{1}{2}(-2) \\ \frac{1}{2}(2) & \frac{1}{2}(-2) & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

characteristic equation,

$$|A - \lambda I| = 0$$

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0$$

$$S_1 = 3 + 3 + 3 = 9$$

$$S_2 = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}$$

$$= (9 - 1) + (9 - 1) + (9 - 1)$$

$$= 8 + 8 + 8$$

$$S_2 = 24$$

$$S_3 = |A| = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 3(9 - 1) - 1(3 + 1) + 1(1 - 3) \\ = 3(8) - 1(4) + 1(-4) \\ = 24 - 4 - 4$$

$$S_3 = 16$$

Eigen Values.

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0.$$

$$\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0.$$

$$\begin{array}{c|cccc} 1 & 1 & -9 & 24 & -16 \\ & 0 & 1 & -8 & 16 \\ \hline 4 & 1 & -8 & 16 & 0 \\ & 0 & 4 & -16 & \\ \hline & 1 & -4 & 0 & \end{array} \quad \begin{array}{l} (C) \div (C) \div \varepsilon \\ (C) \div \varepsilon \quad (C) \div \varepsilon \\ \varepsilon \quad (C) \div \varepsilon \quad (C) \div \varepsilon \end{array}$$

$$\lambda = 1, \lambda = 4, \lambda = -4$$

$$\begin{bmatrix} 1 & 1 & \varepsilon \\ 1 & \varepsilon & 1 \\ \varepsilon & 1 & 1 \end{bmatrix} = A$$

Eigen Values.

$$(A - \lambda I)x = 0.$$

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda = 1$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \textcircled{A}$$

$$2x_1 + x_2 + x_3 = 0 \rightarrow \textcircled{1}$$

$$x_1 + 2x_2 - x_3 = 0 \rightarrow \textcircled{2}$$

$$x_1 - x_2 + 2x_3 = 0 \rightarrow \textcircled{3}$$

$$\frac{x_1}{-1-2} = \frac{x_2}{1+2} = \frac{x_3}{4-1} \quad \leftarrow \quad \frac{x_1}{1-1} = \frac{x_2}{2-1} = \frac{x_3}{2-1}$$

$$\frac{x_1}{-3} = \frac{x_2}{3} = \frac{x_3}{3}$$

$\div 3$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad x = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda = 4$ Sub in (A).

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$$-x_1 + x_2 + x_3 = 0 \rightarrow (4)$$

$$x_1 - x_2 - x_3 = 0 \rightarrow (5)$$

$$x_1 - x_2 - x_3 = 0 \rightarrow (6)$$

From (3),

$$x_1 - x_2 - x_3 = 0.$$

Sub $x_3 = 0$,

$$x_1 - x_2 = 0$$

$$x_1 = x_2$$

$$x_3 = 0, \quad x_1 = 1, \quad x_2 = 1$$

$$x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

case (iii) Let m, n multiply x_1, x_2 .

$$-1 + m + n = 0$$

$$1 + m + 0 = 0.$$

$$x_3 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

$$\frac{\lambda}{1} = \frac{m}{1} = \frac{n}{-1}$$

$$\frac{\lambda}{0-1} = \frac{m}{1-0} = \frac{n}{-1-1}$$

$$\frac{\lambda}{-1} = \frac{m}{1} = \frac{n}{-2}$$

$$x_3 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

$$\text{Matrix} = [x_1, x_2, x_3] = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{bmatrix}$$

Normalised matrix.

$$\begin{aligned} \|x_1\|^2 &= (-1, 1, 1) \cdot (-1, 1, 1) \\ &= 1 + 1 + 1 \\ &= 3 \end{aligned}$$

$$\|x_1\| = \sqrt{3}$$

$$\|x_2\|^2 = (1, 1, 0) \cdot (1, 1, 0) = 1 + 1 + 0 = 2$$

$$\|x_2\| = \sqrt{2}$$

$$\|x_3\|^2 = (-1, 1, -2) \cdot (-1, 1, -2) = 1 + 1 + 4 = 6$$

$$\|x_3\| = \sqrt{6}$$

$$P = \left[\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}, \frac{x_3}{\|x_3\|} \right]$$

$$P = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{bmatrix}, \quad P^T = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{bmatrix}$$

$$AP = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 3 - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} & \frac{3}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 0 & \frac{-3}{\sqrt{6}} + \frac{1}{\sqrt{6}} - \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} + \frac{3}{\sqrt{3}} - \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} + \frac{3}{\sqrt{2}} + 0 & \frac{-1}{\sqrt{6}} + \frac{3}{\sqrt{6}} + \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} - \frac{1}{\sqrt{3}} + \frac{3}{\sqrt{3}} & \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + 0 & \frac{-1}{\sqrt{6}} - \frac{1}{\sqrt{6}} - \frac{6}{\sqrt{6}} \end{bmatrix}$$

$$AP = \begin{bmatrix} \frac{2}{\sqrt{3}} & \frac{4}{\sqrt{2}} & \frac{-4}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{4}{\sqrt{2}} & \frac{4}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-8}{\sqrt{6}} \end{bmatrix}$$

$$D = P^T AP$$

$$= \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{4}{\sqrt{2}} & \frac{-4}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{4}{\sqrt{2}} & \frac{4}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-8}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+1+1}{\sqrt{3}} & \frac{-4+4+0}{\sqrt{6}} & \frac{4+4-8}{\sqrt{2}} \\ \frac{-1+1+0}{\sqrt{6}} & \frac{4+4+0}{\sqrt{2}} & \frac{-4+4+0}{\sqrt{2}} \\ \frac{1+1-2}{\sqrt{8}} & \frac{-4+4+0}{\sqrt{2}} & \frac{4+4+16}{6} \end{bmatrix}$$

$$D = \begin{bmatrix} \frac{3}{3} & 0 & 0 \\ 0 & \frac{8}{2} & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

canonical form.

$$[y_1, y_2, y_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= y_1^2 + 4y_2^2 + 4y_3^2$$

Nature : positive definite.

Rank = 3, Index = 3.

Signature = 3.

It is a positive definite.

③ Reduce the quadratic form $8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 - 8x_2x_3 + 4x_3x_1$ to the canonical form through an orthogonal transformation.

Sol.

$$A = \begin{bmatrix} \text{coeff } x_1^2 & \frac{1}{2} \text{coeff } x_1 x_2 & \frac{1}{2} \text{coeff } x_1 x_3 \\ \frac{1}{2} \text{coeff } x_2 x_1 & \text{coeff } x_2^2 & \frac{1}{2} \text{coeff } x_2 x_3 \\ \frac{1}{2} \text{coeff } x_3 x_1 & \frac{1}{2} \text{coeff } x_3 x_2 & \text{coeff } x_3^2 \end{bmatrix}$$

$$A = \begin{bmatrix} 8 & \frac{1}{2}(-12) & \frac{1}{2}(4) \\ \frac{1}{2}(-12) & 7 & \frac{1}{2}(-8) \\ \frac{1}{2}(4) & \frac{1}{2}(-8) & 3 \end{bmatrix} = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

characteristic Equation

$$|A - \lambda I| = 0.$$

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0.$$

$$S_1 = 8 + 7 + 3 = 18$$

$$S_2 = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$$= (21 - 16) + (24 - 4) + (56 - 36)$$

$$= 5 + 20 + 20.$$

$$S_2 = 45$$

$$S_3 = |A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix} = 8(21 - 16) + 6(-18 + 8) + 2(24 - 14)$$

$$= 8(5) + 6(-10) + 2(10)$$

$$= 40 - 60 + 20$$

$$= 0.$$

$$S_3 = 0.$$

$$\lambda^3 - 18\lambda^2 + 45\lambda - 0 = 0.$$

$$\lambda(\lambda^2 - 18\lambda + 45) = 0.$$

$$\lambda = 0 \quad \lambda^2 - 18\lambda + 45 = 0.$$

$$\lambda - 15 = 0, \quad \lambda - 3 = 0$$

$$\lambda = 15, \quad \lambda = 3$$

$$\begin{array}{r} 45 \\ -15 \cdot 3 \\ \hline 18 \end{array}$$

$$\lambda = 0, 3, 15$$

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

case i) $\lambda = 0$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$8x_1 - 6x_2 + 2x_3 = 0 \rightarrow (1)$$

$$-6x_1 + 7x_2 - 4x_3 = 0 \rightarrow (2)$$

$$2x_1 - 4x_2 + 3x_3 = 0 \rightarrow (3)$$

$$\frac{x_1}{-6 \quad 2} = \frac{x_2}{2 \quad 8} = \frac{x_3}{8 \quad -6}$$

$$\frac{x_1}{24 - 14} = \frac{x_2}{-12 + 32} = \frac{x_3}{56 - 36}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

$\div$ by 10.

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

$$x_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad x_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

case ii) $\lambda = 3$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x_1 - 6x_2 + 2x_3 = 0 \longrightarrow (4)$$

$$-6x_1 + 4x_2 - 4x_3 = 0 \longrightarrow (5)$$

$$2x_1 - 4x_2 \longrightarrow (6)$$

$$\frac{x_1}{-6 \quad 2} = \frac{x_2}{2 \quad 5} = \frac{x_3}{5 \quad -6}$$
$$4 \quad -4 \quad -4 \quad -6 \quad -6 \quad 4$$

$$\frac{x_1}{24-8} = \frac{x_2}{-12+20} = \frac{x_3}{20-36}$$

$$\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16}$$

$\div$ by (8)

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$$x_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Case (iii) $\lambda = 15$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x_1 - 6x_2 + 2x_3 = 0 \longrightarrow (7)$$

$$-6x_1 - 8x_2 - 4x_3 = 0 \longrightarrow (8)$$

$$2x_1 - 4x_2 - 12x_3 = 0 \longrightarrow (9)$$

$$\frac{x_1}{-6 \quad 2 \quad -8 \quad -4} = \frac{x_2}{2 \quad -7 \quad -4 \quad -6} = \frac{x_3}{-7 \quad -6 \quad -6 \quad -8}$$

$$\frac{x_1}{24+16} = \frac{x_2}{-12-28} = \frac{x_3}{56-36}$$

$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20}$$

÷ by 20.

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$x_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Normalised Matrix.

$$\|x_1\|^2 = (1, 2, 2) (1, 2, 2)$$

$$= 1+4+4$$

$$= 9$$

$$\|x_1\| = \sqrt{9} = 3$$

$$\|x_2\|^2 = (2, 1, -2) (2, 1, -2)$$

$$= 4+1+4$$

$$= 9$$

$$\|x_2\| = \sqrt{9} = 3$$

$$\|x_3\|^2 = (2, -2, 1) (2, -2, 1) = (4+4+1) = 9$$

$$\|x_3\| = \sqrt{9} = 3$$

$$P = \left[\frac{x_1}{\|x_1\|} \quad \frac{x_2}{\|x_2\|} \quad \frac{x_3}{\|x_3\|} \right]$$

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \quad P^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$AP = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{8-12+4}{3} & \frac{16-6-4}{3} & \frac{16+12+2}{3} \\ \frac{16-6+14-8}{3} & \frac{-12+7+8}{3} & \frac{-12-14-4}{3} \\ \frac{2-8+6}{3} & \frac{4-4-6}{3} & \frac{4+8+3}{3} \end{bmatrix}$$

$$AP = \begin{bmatrix} 0 & 2 & 10 \\ 0 & 1 & -10 \\ 0 & -2 & 5 \end{bmatrix}$$

$$D = P^T AP$$

$$D = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 10 \\ 0 & 1 & -10 \\ 0 & -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{0+0+0}{3} & \frac{2+2-4}{3} & \frac{10-20+10}{3} \\ 0+0+0 & \frac{4+1+4}{3} & \frac{20-10-10}{3} \\ 0+0+0 & \frac{4-2-2}{3} & \frac{20+20+5}{3} \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

canonical form.

$$[y_1, y_2, y_3] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 0y_1^2 + 3y_2^2 + 15y_3^2$$

Nature : positive Matrix

Rank : 2

Index : 2

Signature : $2 - 0 = 2$ (+) - (-ve).

It is a positive definite.

4) Reduce the quadratic form $3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_2x_3 + 2x_3x_1 - 2x_1x_2$ to a canonical form by orthogonal reduction:

Sol

$$A = \begin{bmatrix} \text{Coeff } x_1^2 & \frac{1}{2} \text{Coeff } x_1x_2 & \frac{1}{2} \text{Coeff } x_1x_3 \\ \frac{1}{2} \text{Coeff } x_2x_1 & \text{Coeff } x_2^2 & \frac{1}{2} \text{Coeff } x_2x_3 \\ \frac{1}{2} \text{Coeff } x_3x_1 & \frac{1}{2} \text{Coeff } x_3x_2 & \text{Coeff } x_3^2 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & \frac{1}{2}(-2) & \frac{1}{2}(2) \\ \frac{1}{2}(-2) & 5 & \frac{1}{2}(-2) \\ \frac{1}{2}(2) & \frac{1}{2}(-2) & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

characteristic Equation:

$$|A - \lambda I| = 0$$

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$$

$$s_1 = 3 + 5 + 3 = 11$$

$$s_2 = \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$

$$= (15 - 1) + (9 - 1) + (15 - 1)$$

$$= 14 + 8 + 14$$

$$s_2 = 36$$

$$s_3 = |A| = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 3(15 - 1) + 1(-3 + 1) + 1(1 - 5)$$

$$= 3(14) + 1(-2) + 1(-4)$$

$$= 42 - 2 - 4$$

$$= 36$$

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\begin{array}{l} 2 \left| \begin{array}{cccc} 1 & -11 & 36 & -36 \\ 0 & 2 & -18 & 36 \end{array} \right. \\ 3 \left| \begin{array}{ccc|c} 1 & -9 & 18 & 0 \\ 0 & 3 & -18 & \\ \hline 1 & -6 & 0 & \end{array} \right. \end{array}$$

$$\lambda = 2, \lambda = 3, \lambda - 6 = 0$$

$$\lambda = 6$$

Eigen value:

$$|A - \lambda I| x = 0.$$

$$\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda = 2.$$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \textcircled{A}$$

$$x_1 - x_2 + x_3 = 0 \rightarrow \textcircled{1}$$

$$-x_1 + 3x_2 - x_3 = 0 \rightarrow \textcircled{2}$$

$$x_1 - x_2 + x_3 = 0 \rightarrow \textcircled{3}$$

$$\frac{x_1}{-1+1} = \frac{x_2}{+1-1} = \frac{x_3}{1-1}$$

$$\frac{x_1}{1-3} = \frac{x_2}{-1+1} = \frac{x_3}{3-1}$$

$$\frac{x_1}{-4} = \frac{x_2}{0} = \frac{x_3}{2}$$

÷ by 2

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$$x_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad x_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda = 3$$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_2 + x_3 = 0 \rightarrow \textcircled{4}$$

$$-x_1 + 2x_2 - x_3 = 0 \rightarrow \textcircled{5}$$

$$x_1 - x_2 = 0 \rightarrow \textcircled{6}$$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{0} = 1$$

$$\frac{x_1}{1-2} = \frac{x_2}{-1+0} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

÷ by (-)

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$x_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda = 6.$$

$$\begin{bmatrix} -3 & +1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3x_1 - x_2 + x_3 = 0 \rightarrow \textcircled{7}$$

$$-x_1 - x_2 - x_3 = 0 \rightarrow \textcircled{8}$$

$$x_1 - x_2 - 3x_3 = 0 \rightarrow \textcircled{9}$$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{-3}$$

$$\frac{x_1}{1+1} = \frac{x_2}{-1-3} = \frac{x_3}{3-1}$$

$$\frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2}$$

div by 2.

$$\frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$x_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, x_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Normalised matrix.

$$\|x_1\|^2 = (-1, 0, 1) \cdot (-1, 0, 1) = 1 + 0 + 1 = 2$$

$$\|x_1\| = \sqrt{2}$$

$$\|x_1\|^2 = (1, 1, 1) \cdot (1, 1, 1) = 1 + 1 + 1 = 3$$

$$\|x_1\| = \sqrt{3}$$

$$\|x_2\|^2 = (1, -2, 1) \cdot (1, -2, 1) = 1 + 4 + 1 = 6$$

$$\|x_2\| = \sqrt{6}$$

$$P = \left[\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}, \frac{x_3}{\|x_3\|} \right]$$

$$P = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} \quad P^T = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$AP = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-3+0+1}{\sqrt{2}} & \frac{3-1+1}{\sqrt{3}} & \frac{3+2+1}{\sqrt{6}} \\ \frac{1+0-1}{\sqrt{2}} & \frac{-1+5-1}{\sqrt{3}} & \frac{-1+10-1}{\sqrt{6}} \\ \frac{-1+0+3}{\sqrt{2}} & \frac{1-1+3}{\sqrt{3}} & \frac{1+2+3}{\sqrt{6}} \end{bmatrix}$$

$$AP = \begin{bmatrix} \frac{-2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}} \\ 0 & \frac{3}{\sqrt{3}} & \frac{-12}{\sqrt{6}} \\ \frac{2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}} \end{bmatrix}$$

$$D = P^T A P$$

$$D = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} -2/\sqrt{2} & 3/\sqrt{3} & 6/\sqrt{6} \\ 0 & 3/\sqrt{3} & -12/\sqrt{6} \\ 2/\sqrt{2} & 3/\sqrt{3} & 6/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2+0+2}{\sqrt{2}} & \frac{-3+0+3}{\sqrt{6}} & \frac{-6+0+6}{\sqrt{12}} \\ \frac{-2+0+2}{\sqrt{6}} & \frac{3+3+3}{3} & \frac{6-12+6}{\sqrt{18}} \\ \frac{-2+0+2}{\sqrt{12}} & \frac{3+6+3}{\sqrt{18}} & \frac{6+24+6}{6} \end{bmatrix}$$

$$D = \begin{bmatrix} \frac{4}{2} & 0 & 0 \\ 0 & 9/3 & 0 \\ 0 & 0 & 36/6 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Canonical form.

$$[y_1, y_2, y_3] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0$$

$$2y_1^2 + 3y_2^2 + 6y_3^2 = 0$$

Nature: positive matrix

Rank: 3

Index: 3

Signature = 3

It is a positive definite

- 5) Reduce the quadratic form $2x_1x_2 - 2x_2x_3 + 2x_3x_1$ into the canonical form and hence find its nature.

Sol

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\lambda = 1, 1, -2$$

canonical form

$$y_1^2 + y_2^2 - 2y_3^2 = 0$$

positive index = 2

negative index = 1

rank = 3

It is indefinite matrix.

we want to put the sum, same model only.

25/2/26

QR Decomposition

$$R = Q^T A$$

$$A = QR$$

Gram - Schmidt process

$$v_1 = u_1$$

$$v_2 = u_2 - \frac{(u_2, v_1)}{\|v_1\|^2} v_1$$

$$v_3 = u_3 - \frac{(u_3, v_1)}{\|v_1\|^2} v_1 - \frac{(u_3, v_2)}{\|v_2\|^2} v_2$$

orthogonal set = (V_1, V_2, V_3) .

orthonormal basis = $Q = \left[\frac{V_1}{\|V_1\|}, \frac{V_2}{\|V_2\|}, \frac{V_3}{\|V_3\|} \right]$

upper triangular matrix $\Rightarrow R = Q^T A$

$$A = QR.$$

Exms

1. Find the QR decomposition of the matrix $A = \begin{bmatrix} -4 & 2 & 2 \\ 3 & -3 & 3 \\ 6 & 6 & 0 \end{bmatrix}$

sol

$$A = \begin{bmatrix} -4 & 2 & 2 \\ 3 & -3 & 3 \\ 6 & 6 & 0 \end{bmatrix}$$

Let u_1, u_2, u_3 be the columns of A .

$$u_1 = \begin{bmatrix} -4 \\ 3 \\ 6 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

$$\text{Let } v_1 = u_1 = \begin{bmatrix} -4 \\ 3 \\ 6 \end{bmatrix}$$

$$\begin{aligned} \|v_1\|^2 &= (v_1, v_1) = (-4, 3, 6) \cdot (-4, 3, 6) \\ &= 16 + 9 + 36 \end{aligned}$$

$$\|v_1\|^2 = 61$$

$$\|v_1\| = \sqrt{61}$$

$$V_2 = u_2 - \frac{(u_2, v_1)}{\|v_1\|^2} v_1$$

$$(u_2, v_1) = (2, -3, 6) \cdot (-4, 3, 6)$$

$$= -8 - 9 + 36$$

$$= -17 + 36$$

$$(u_2, v_1) = 19$$

$$V_2 = (2, -3, 6) - \frac{19}{61} (-4, 3, 6)$$

$$= (2, -3, 6) - 0.31 (-4, 3, 6)$$

$$= (2, -3, 6) + (1.24, -0.93, -1.86)$$

$$= (2, -3, 6) + (1.24, -0.93, -1.86)$$

$$= (2 + 1.24, -3 - 0.93, 6 - 1.86)$$

$$V_2 = (3.24, -3.93, 4.14)$$

$$\|V_2\|^2 = (3.24, -3.93, 4.14) \cdot (3.24, -3.93, 4.14)$$

$$= 10.5 + 15.4 + 17.14$$

$$= 43.08$$

$$\|V_2\| = \sqrt{43.08}$$

$$\|V_2\| = 6.6$$

$$V_3 = u_3 - \frac{(u_3, v_1)}{\|v_1\|^2} v_1 - \frac{(u_3, v_2)}{\|v_2\|^2} v_2$$

$$(u_3, v_1) = (2, 3, 6) \cdot (-4, 3, 6)$$

$$= -8 + 9 + 0$$

$$(u_3, v_1) = 1$$

$$\left[\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right] = B$$

$$(u_3, v_2) = (2, 3, 0) \cdot (3, 24, -3.93, 4.14)$$

$$= 6.48 - 11.79 - 0$$

$$(u_3, v_2) = -5.31$$

$$v_3 = (2, 3, 0) - \frac{1}{61}(-4, 3, 6) - \frac{-5.31}{43.08}(3.24, -3.93, 4.14)$$

$$= (2, 3, 0) - 0.01(-4, 3, 6) + 0.12(3.24, -3.93, 4.14)$$

$$= (2, 3, 0) + (0.04, -0.03, -0.06) + (0.38, -0.47, 0.49)$$

$$= (2 + 0.04 + 0.38, 3 - 0.03 - 0.47, 0 - 0.06 + 0.49)$$

$$v_3 = (2.42, 2.5, 0.43)$$

$$\|v_3\|^2 = (v_3, v_3)$$

$$= (2.42, 2.5, 0.43) \cdot (2.42, 2.5, 0.43)$$

$$= 5.85 + 6.25 + 0.18$$

$$= 12.28$$

$$\|v_3\| = \sqrt{12.28}$$

$$\|v_1\| = \sqrt{61} = 7.8$$

$$\|v_2\| = \sqrt{43.08} = 6.5$$

$$\|v_3\| = \sqrt{12.28} = 3.5$$

orthogonal sets $= [v_1, v_2, v_3] = \begin{bmatrix} -4 & 3.24 & 2.42 \\ 3 & -3.93 & 2.5 \\ 6 & 4.14 & 0.43 \end{bmatrix}$

orthogonal basis

$$a = \left[\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right]$$

$$Q = \begin{bmatrix} \frac{-4}{7.8} & \frac{3.24}{6.5} & \frac{2.42}{3.5} \\ \frac{3}{7.8} & \frac{-3.93}{6.5} & \frac{2.5}{3.5} \\ \frac{6}{7.8} & \frac{4.14}{6.5} & \frac{0.43}{3.5} \end{bmatrix}$$

$$Q = \begin{bmatrix} -0.5 & 0.4 & 0.6 \\ 0.3 & -0.6 & 0.7 \\ 0.7 & 0.6 & 0.12 \end{bmatrix} \Rightarrow Q' = \begin{bmatrix} -0.5 & 0.3 & 0.7 \\ 0.4 & -0.6 & 0.6 \\ 0.6 & 0.7 & 0.12 \end{bmatrix}$$

$$R = Q'A$$

$$R = \begin{bmatrix} -0.5 & 0.3 & 0.7 \\ 0.4 & -0.6 & 0.6 \\ 0.6 & 0.7 & 0.12 \end{bmatrix} \begin{bmatrix} -4 & 2 & 2 \\ 3 & -3 & 3 \\ 6 & 6 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 2 + 0.9 + 4.2 & -1 - 0.9 + 4.2 & -1 + 0.9 + 0 \\ -1.6 - 1.8 + 3.6 & 0.8 + 1.8 + 3.6 & 0.8 - 1.8 + 0 \\ -2.4 + 2.1 + 0.72 & 1.2 - 2.1 + 0.72 & 1.2 + 2.1 + 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 7.1 & 2.3 & -0.1 \\ 0.2 & 6.2 & -1 \\ 0.4 & -0.18 & 3.3 \end{bmatrix}$$

$$R = \begin{bmatrix} 7.1 & 2.3 & -0.1 \\ 0 & 6.2 & -1 \\ 0 & 0 & 3.3 \end{bmatrix}$$

$$A = QR$$

$$= \begin{bmatrix} -0.5 & 0.4 & 0.6 \\ 0.3 & -0.6 & 0.7 \\ 0.7 & 0.6 & 0.12 \end{bmatrix} \begin{bmatrix} 7.1 & 2.3 & -0.1 \\ 0 & 6.2 & -1 \\ 0 & 0 & 3.3 \end{bmatrix}$$

2) Find the QR decomposition of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Sol

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Let u_1, u_2, u_3 be the column of A .

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$u_1 = v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \|u_1\|^2 &= (u_1, u_1) \\ &= (1, 1, 1, 1) \cdot (1, 1, 1, 1) \\ &= 1 + 1 + 1 + 1 \\ &= 4 \end{aligned}$$

$$\|u_1\| = \sqrt{4}$$

$$v_2 = u_2 - \frac{(u_2, v_1)}{\|v_1\|^2} v_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$(u_1, v_1) = (0, 1, 1, 1) (1, 1, 1, 1)$$

$$= 0 + 1 + 1 + 1$$

$$= 3$$

$$v_2 = (0, 1, 1, 1) - \frac{3}{4} (1, 1, 1, 1)$$

$$= (0, 1, 1, 1) - (0.75, 0.75, 0.75, 0.75)$$

$$= (0, 1, 1, 1) - (0.75, 0.25, 0.25, 0.25)$$

$$v_2 = (-0.75, 0.25, 0.25, 0.25)$$

$$\|v_2\|^2 = (v_2, v_2)$$

$$= (-0.75, 0.25, 0.25, 0.25) (-0.75, 0.25, 0.25, 0.25)$$

$$= -1/5 \quad 0.56 + 0.06 + 0.06 + 0.06$$

$$= 0.74$$

$$\|v_2\| = \sqrt{0.74}$$

$$v_3 = u_3 - \frac{(u_3, v_1)}{\|v_1\|^2} v_1 - \frac{(u_3, v_2)}{\|v_2\|^2} v_2$$

$$(u_3, v_1) = (0, 0, 1, 1) (1, 1, 1, 1)$$

$$= 0 + 0 + 1 + 1$$

$$= 2$$

$$(u_3, v_2) = (0, 0, 1, 1) (-0.75, 0.25, 0.25, 0.25)$$

$$= 0 + 0 + 0.25 + 0.25$$

$$= 0.5$$

$$V_3 = (0, 0, 1, 1) - \frac{2}{4} (1, 1, 1, 1) - \frac{0.5}{0.74} (-0.75, 0.25, 0.25, 0.25)$$

$$= (0, 0, 1, 1) - 0.5 (1, 1, 1, 1) - 0.6 (-0.75, 0.25, 0.25, 0.25)$$

$$= (0, 0, 1, 1) - (0.5, 0.5, 0.5, 0.5) + (0.45, -0.15, -0.15, -0.15)$$

$$= (-0.05, -0.65, 0.35, 0.35)$$

$$= (0 - 0.5 + 0.45) (0 - 0.5 - 0.15) (-0.5 - 0.15)$$

$$V_3 = (-0.05, -0.65, 0.35, 0.35)$$

$$\|V_3\|^2 = (V_3, V_3)$$

$$= (-0.05, -0.65, 0.35, 0.35) \cdot (-0.05, -0.65, 0.35, 0.35)$$

$$= 0.0025 + 0.4225 + 0.1225 + 0.1225$$

$$= 0.67$$

$$\|V_3\| = \sqrt{0.67} = 0.81$$

$$\|V_1\| = \sqrt{4} = 2$$

$$\|V_2\| = \sqrt{0.74} = 0.86$$

$$\|V_3\| = \sqrt{0.67} = 0.81$$

orthogonal sets $= [V_1, V_2, V_3] = \begin{bmatrix} 1 & -0.75 & -0.05 \\ 1 & 0.25 & -0.65 \\ 1 & 0.25 & 0.35 \\ 1 & 0.25 & 0.35 \end{bmatrix}$

orthogonal basis.

$$Q = \left[\frac{V_1}{\|V_1\|} \quad \frac{V_2}{\|V_2\|} \quad \frac{V_3}{\|V_3\|} \right]$$

$$Q = \begin{bmatrix} \frac{1}{2} & \frac{-0.75}{0.86} & \frac{-0.05}{0.81} \\ \frac{1}{2} & \frac{0.25}{0.86} & \frac{-0.65}{0.81} \\ \frac{1}{2} & \frac{0.25}{0.86} & \frac{0.35}{0.81} \\ \frac{1}{2} & \frac{0.25}{0.86} & \frac{0.35}{0.81} \end{bmatrix}$$

$$Q = \begin{bmatrix} 0.5 & -0.87 & -0.06 \\ 0.5 & 0.29 & -0.80 \\ 0.5 & 0.29 & 0.43 \\ 0.5 & 0.29 & 0.43 \end{bmatrix} = Q = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ -0.87 & 0.29 & 0.29 & 0.29 \\ -0.06 & -0.80 & 0.43 & 0.43 \end{bmatrix}$$

$$R = Q^T A$$

$$R = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ -0.87 & 0.29 & 0.29 & 0.29 \\ -0.06 & -0.80 & 0.43 & 0.43 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5+0.5+0.5+0.5 & 0+0.5+0.5+0.5 & 0+0+0.5+0.5 \\ -0.87+(0.29)+(0.29)+(0.29) & 0+0.29+0.29+0.29 & 0+0+0.29+0.29 \\ -0.06+(-0.80)+0.43+0.43 & 0+(-0.80)+0.43+0.43 & 0+0+0.43+0.43 \end{bmatrix}$$

$$R = \begin{bmatrix} 2 & 1.5 & 1 \\ 0 & 0.87 & 0.86 \\ 0 & 0.06 & 0.86 \end{bmatrix}$$

$$R = \begin{bmatrix} 2 & 1.5 & 1 \\ 0 & 0.87 & 0.58 \\ 0 & 0 & 0.86 \end{bmatrix}$$

$$A = QR$$

$$= \begin{bmatrix} 0.5 & -0.87 & -0.06 \\ 0.5 & 0.29 & -0.80 \\ 0.5 & 0.29 & 0.43 \\ 0.5 & 0.29 & 0.43 \end{bmatrix} \begin{bmatrix} 2 & 1.5 & 1 \\ 0 & 0.87 & 0.58 \\ 0 & 0 & 0.86 \end{bmatrix}$$

3) Find the QR decomposition of $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

sol

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$v_1 = u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\|v_1\|^2 = (v_1, v_1)$$

$$= (1, 0, 0) (1, 0, 0)$$

$$= 1$$

$$\|v_1\| = \sqrt{1}$$

$$v_2 = u_2 - \frac{(u_2, v_1)}{\|v_1\|^2} v_1$$

$$(u_2, v_1) = (0, 2, -1) \cdot (1, 0, 0)$$

$$= 0 + 0 + 0$$

$$= 0$$

$$v_2 = (0, 2, -1) - \frac{0}{1} \cdot (1, 0, 0)$$

$$= (0, 2, -1) - 0$$

$$v_2 = (0, 2, -1)$$

$$\|v_2\|^2 = (v_2, v_2)$$

$$= (0, 2, -1) \cdot (0, 2, -1)$$

$$= 0 + 4 + 1$$

$$= 5$$

$$\|v_2\| = \sqrt{5}$$

$$v_3 = u_3 - \frac{(u_3, v_1)}{\|v_1\|^2} v_1 - \frac{(u_3, v_2)}{\|v_2\|^2} v_2$$

$$(u_3, v_1) = (2, 0, 1) \cdot (1, 0, 0)$$

$$= 2 + 0 + 0$$

$$(u_3, v_1) = 2$$

$$(u_3, v_2) = (2, 0, 1) \cdot (0, 2, -1)$$

$$= 0 + 0 - 1$$

$$(u_3, v_2) = -1$$

$$v_3 = (2, 0, 1) - \frac{2}{1} (1, 0, 0) + \frac{1}{5} (0, 2, -1)$$

$$= (2, 0, 1) - (2, 0, 0) + \left(0, \frac{2}{5}, -\frac{1}{5}\right)$$

$$= (2, 0, 1) + (-2, 0, 0) + (0, \frac{2}{5}, -\frac{1}{5})$$

$$= (2-2+0) (0-0+\frac{2}{5}) + (1-0-\frac{1}{5})$$

$$= (0, \frac{2}{5}, 1-\frac{1}{5})$$

$$= (0, \frac{2}{5}, \frac{4}{5})$$

$$V_3 = \frac{2}{5} (0, 1, 2)$$

$$V_3 = (0, 1, 2)$$

$$\|V_3\|^2 = (V_3, V_3)$$

$$= (0, 1, 2) (0, 1, 2)$$

$$= (0+1+4)$$

$$= 5$$

$$\|V_3\| = \sqrt{5}$$

$$\text{orthogonal sets} = (V_1, V_2, V_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\text{orthog normal basis } Q = \left[\frac{V_1}{\|V_1\|}, \frac{V_2}{\|V_2\|}, \frac{V_3}{\|V_3\|} \right]$$

$$Q = \begin{bmatrix} \frac{1}{1} & \frac{0}{\sqrt{5}} & \frac{0}{\sqrt{5}} \\ \frac{0}{1} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{0}{1} & \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 0 & \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$R = Q^T A$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2/\sqrt{5} & -1/\sqrt{5} \\ 0 & 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+0 & 2+0+0 \\ 0+0+0 & 0+4/\sqrt{5}+1/\sqrt{5} & 0+0-1/\sqrt{5} \\ 0+0+0 & 0+2/\sqrt{5}-2/\sqrt{5} & 0+0+2/\sqrt{5} \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5/\sqrt{5} & -1/\sqrt{5} \\ 0 & 0 & 2/\sqrt{5} \end{bmatrix}$$

$$A = QR$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2/\sqrt{5} & 1/\sqrt{5} \\ 0 & -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5/\sqrt{5} & -1/\sqrt{5} \\ 0 & 0 & 2/\sqrt{5} \end{bmatrix}$$

4) Find the QR decomposition of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

sol

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad u_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad u_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

$$v_1 = u_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \|v_1\|^2 &= (v_1, v_1) \\ &= (0, 0, -1, 0) (0, 0, -1, 0) \end{aligned}$$

$$= 0 + 0 + 1 + 0$$

$$= 1$$

$$\|v_1\| = \sqrt{1} = 1$$

$$v_2 = u_2 - \frac{(u_2, v_1)}{\|v_1\|^2} v_1$$

$$(u_2, v_1) = (1, 1, 0, -1) (0, 0, -1, 0)$$

$$= 0 + 0 + 0 + 0$$

$$= 0$$

$$v_2 = (1, 1, 0, -1) - \frac{0}{1} (0, 0, -1, 0)$$

$$= (1, 1, 0, -1) - 0$$

$$v_2 = (1, 1, 0, -1)$$

$$\|v_2\|^2 = (v_2, v_2)$$

$$= (1, 1, 0, -1) (1, 1, 0, -1)$$

$$= 1 + 1 + 0 + 1$$

$$= 3$$

$$\|v_2\| = \sqrt{3}$$

$$v_3 = u_3 - \frac{(u_3, v_1)}{\|v_1\|^2} v_1 - \frac{(u_3, v_2)}{\|v_2\|^2} v_2$$

$$(u_3, v_1) = (0, 2, 1, 1) \cdot (0, 0, -1, 0)$$

$$= 0 + 0 + 1 + 0$$

$$= -1$$

$$(u_3, v_2) = (0, 2, 1, 1) \cdot (1, 1, 0, -1)$$

$$= 0 + 2 + 0 + 1$$

$$= 3$$

$$v_3 = (0, 2, 1, 1) - \frac{-1}{1} (0, 0, -1, 0) - \frac{3}{3} (1, 1, 0, -1)$$

$$= (0, 2, 1, 1) + (0, 0, 1, 0) + (-1, -1, 0, 1)$$

$$= (0 + 0 - 1, 2 + 0 - 1, 1 + 1 + 0, 1 + 0 + 1)$$

$$= (-1, 1, 2, 2) = (-1/3, 5/3, 0, 4/3)$$

$$= 1/3 (-1, 5, 0, 4)$$

$$\|v_3\|^2 = (v_3, v_3)$$

$$= (-1, 5, 0, 4) \cdot (-1, 5, 0, 4)$$

$$= 1 + 25 + 0 + 16$$

$$= 42$$

$$\|v_3\| = \sqrt{42}$$

$$\text{orthogonal sets } (v_1, v_2, v_3) = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 5 \\ -1 & 0 & 2 \\ 0 & -1 & 4 \end{bmatrix}$$

orthonormal basis

$$Q = \left[\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right]$$

$$\begin{aligned}
 \|v_1\| &= 1 \\
 \|v_2\| &= \sqrt{3} \\
 \|v_3\| &= \sqrt{42}
 \end{aligned}
 \quad Q = \begin{bmatrix} 0 & 1/\sqrt{3} & -1/\sqrt{42} \\ 0 & 1/\sqrt{3} & 5/\sqrt{42} \\ -1 & 0 & 0 \\ 0 & -1/\sqrt{3} & 4/\sqrt{42} \end{bmatrix}$$

$$R = Q^T A$$

$$= \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & 0 & -1/\sqrt{3} \\ -1/\sqrt{42} & 5/\sqrt{42} & 0 & 4/\sqrt{42} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+1+0 & 0+0+0-0 & 0+0+1+0 \\ 0+0+0+0 & 1/\sqrt{3}+1/\sqrt{3}+0+1/\sqrt{3} & 0+2/\sqrt{3}+0-1/\sqrt{3} \\ 0+0+0+0 & -1/\sqrt{42}+5/\sqrt{42}+0-4/\sqrt{42} & 0+10/\sqrt{42}+0+4/\sqrt{42} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3/\sqrt{3} & 1/\sqrt{3} \\ 0 & 0 & 14/\sqrt{42} \end{bmatrix}$$

$$A = QR$$

$$= \begin{bmatrix} 0 & 1/\sqrt{3} & -1/\sqrt{42} \\ 0 & 1/\sqrt{3} & 5/\sqrt{42} \\ -1 & 0 & 0 \\ 0 & -1/\sqrt{3} & 4/\sqrt{42} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3/\sqrt{3} & 1/\sqrt{3} \\ 0 & 0 & 14/\sqrt{42} \end{bmatrix}$$

5) Find the QR decomposition of the matrix $A = \begin{bmatrix} 4 & 3 & -2 \\ 3 & 1 & -1 \\ -2 & -1 & 0 \end{bmatrix}$

Sol

$$A = \begin{bmatrix} 4 & 3 & -2 \\ 3 & 1 & -1 \\ -2 & -1 & 0 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$$

$$v_1 = u_1 = \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}$$

$$\|v_1\|^2 = (v_1, v_1)$$

$$= (4, 3, -2) (4, 3, -2)$$

$$= 16 + 9 + 4$$

$$= 29$$

$$\|v_1\| = \sqrt{29}$$

$$v_2 = u_2 - \frac{(u_2, v_1)}{\|v_1\|^2} v_1$$

$$(u_2, v_1) = (3, 1, -1) (4, 3, -2)$$

$$= 12 + 3 + 2$$

$$= 17$$

$$v_2 = (3, 1, -1) - \frac{17}{29} (4, 3, -2)$$

$$= (3, 1, -1) - 0.58 (4, 3, -2)$$

$$= (3, 1, -1) + (-2.32, -1.74, +1.16)$$

$$= (3 - 2.32) (1 - 1.74) (-1 + 1.16)$$

$$= (0.68, -0.74, 0.16)$$

$$\|V_2\|^2 = (V_2, V_2)$$

$$= (0.68, -0.74, 0.16) (0.68, -0.74, 0.16)$$

$$= 0.46 + 0.54 + 0.02$$

$$\|V_2\|^2 = 1.02$$

$$\|V_2\| = \sqrt{1.02}$$

$$V_3 = u_3 - \frac{(u_3, V_1)}{\|V_1\|^2} V_1 - \frac{(u_3, V_2)}{\|V_2\|^2} V_2$$

$$(u_3, V_1) = (-2, -1, 0) (4, 3, -2)$$

$$= -8 - 3 + 0$$

$$= -11$$

$$(u_3, V_2) = (-2, -1, 0) (0.68, -0.74, 0.16)$$

$$= -1.36 + 0.74 + 0$$

$$= -0.62$$

$$V_3 = (-2, -1, 0) - \frac{-11}{29} (4, 3, -2) - \frac{-0.62}{1.02} (0.68, -0.74, 0.16)$$

$$= (-2, -1, 0) + 0.37 (4, 3, -2) + 0.60 (0.68, -0.74, 0.16)$$

$$= (-2, -1, 0) + (1.48, 1.11, -0.74) + (0.40, -0.44, 0.09)$$

$$= (-2 + 1.48 + 0.40) (-1 + 1.11, -0.44) (0, -0.74 + 0.09)$$

$$V_3 = (-0.12, -0.33, -0.65)$$

$$\|V_3\|^2 = (V_3, V_3)$$

$$= (-0.12, -0.33, -0.65) (-0.12, -0.33, -0.65)$$

$$= 0.01 + 0.11 + 0.42$$

$$= 0.53$$

$$\|v_3\| = \sqrt{0.53}, \quad \|v_1\| = \sqrt{29} = 5.3, \quad \|v_3\| = \sqrt{0.53}$$

$$\|v_2\| = \sqrt{1.02} = 1.0, \quad \dots = 0.7$$

orthogonal sets = $[v_1, v_2, v_3] = \begin{bmatrix} 4 & 0.68 & -0.12 \\ 3 & -0.74 & -0.33 \\ -2 & 0.16 & -0.65 \end{bmatrix}$

orthogonal

orthonormal basis $Q = \left[\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right]$

$$Q = \begin{bmatrix} \frac{4}{\sqrt{29}} & \frac{0.68}{\sqrt{1.02}} & \frac{-0.12}{\sqrt{0.53}} \\ \frac{3}{\sqrt{29}} & \frac{-0.74}{\sqrt{1.02}} & \frac{-0.33}{\sqrt{0.53}} \\ \frac{-2}{\sqrt{29}} & \frac{0.16}{\sqrt{1.02}} & \frac{-0.65}{\sqrt{0.53}} \end{bmatrix} \quad Q = \begin{bmatrix} \frac{4}{5.3} & \frac{0.68}{1.0} & \frac{-0.12}{0.7} \\ \frac{3}{5.3} & \frac{-0.74}{1.0} & \frac{-0.33}{0.7} \\ \frac{-2}{5.3} & \frac{0.16}{1.0} & \frac{-0.65}{0.7} \end{bmatrix}$$

$R \neq Q^T A$

$$Q = \begin{bmatrix} 0.75 & 0.68 & -0.17 \\ 0.56 & -0.74 & -0.47 \\ \frac{-4.037}{-0.37} & 0.16 & -0.92 \end{bmatrix} \quad Q^T = \begin{bmatrix} 0.75 & 0.56 & -0.37 \\ 0.68 & -0.74 & 0.16 \\ -0.17 & -0.47 & -0.92 \end{bmatrix}$$

$R = Q^T A$

$$= \begin{bmatrix} 0.75 & 0.56 & -0.37 \\ 0.68 & -0.74 & 0.16 \\ -0.17 & -0.47 & -0.92 \end{bmatrix} \begin{bmatrix} 4 & 3 & -2 \\ 3 & 1 & -1 \\ -2 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3+1.68+0.74 & 2.25+0.56+0.37 & -1.5-0.56+0 \\ 2.72+2.22-0.32 & 2.04-0.74-0.16 & -1.36+0.74+0 \\ -0.68-1.41+1.84 & -0.51-0.47+0.92 & 0.34+0.47+0 \end{bmatrix}$$

$$R = \begin{bmatrix} 5.42 & 3.18 & -2.06 \\ 0.18 & 1.14 & -0.62 \\ \cancel{4.62} & & \\ -0.25 & -0.06 & 0.81 \end{bmatrix}$$

$$R = \begin{bmatrix} 5.42 & 3.18 & -2.06 \\ 0 & 1.14 & -0.62 \\ 0 & 0 & 0.81 \end{bmatrix}$$

$$A = QR$$

$$A = \begin{bmatrix} 0.75 & 0.68 & -0.17 \\ 0.56 & -0.74 & -0.47 \\ -0.37 & 0.16 & -0.92 \end{bmatrix} \begin{bmatrix} 5.42 & 3.18 & -2.06 \\ 0 & 1.14 & -0.62 \\ 0 & 0 & 0.81 \end{bmatrix}$$

6) Find the QR decomposition of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Sol

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Let

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$v_1 = u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \|v_1\|^2 &= (v_1, v_1) \\ &= (1, 1, 1) (1, 1, 1) \\ &= 1 + 1 + 1 \\ &= 3 \end{aligned}$$

$$\|v_1\| = \sqrt{3}$$

$$v_2 = u_2 - \frac{(u_2, v_1)}{\|v_1\|^2} v_1$$

$$(u_2, v_1) = (0, 1, 1) \cdot (1, 1, 1)$$

$$= 0 + 1 + 1$$

$$= 2$$

$$v_2 = (0, 1, 1) - \frac{2}{3} (1, 1, 1)$$

$$= (0, 1, 1) - 0.6(1, 1, 1) = (0, 1, 1) - \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$= (0, 1, 1) + (-0.6, -0.6, -0.6) = \left(0 - \frac{2}{3}, 1 - \frac{2}{3}, 1 - \frac{2}{3}\right)$$

$$= (0 - 0.6) (1 - 0.6) (1 - 0.6) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$v_2 = (-0.6, 0.4, 0.4) = \frac{1}{3} (-2, 1, 1) = (-2, 1, 1)$$

$$v_2 = (-2, 1, 1)$$

$$\|v_2\|^2 = (v_2, v_2)$$

$$= (-2, 1, 1) \cdot (-2, 1, 1) = (-0.6, 0.4, 0.4) \cdot (-0.6, 0.4, 0.4)$$

$$= 0.36 + 0.16 + 0.16 = 4 + 1 + 1$$

$$= 6$$

$$= 0.68$$

$$\|v_2\| = \sqrt{6}$$

$$\|v_2\| = \sqrt{0.68}$$

$$v_3 = u_3 - \frac{(u_3, v_1)}{\|v_1\|^2} v_1 - \frac{(u_3, v_2)}{\|v_2\|^2} v_2$$

$$(u_3, v_1) = (0, 0, 1) \cdot (1, 1, 1)$$

$$= 0 + 0 + 1$$

$$= 1$$

$$(u_3, v_2) = (0, 0, 1) \cdot (-2, 1, 1)$$

$$= 0 + 0 + 0.4 = 0 + 0 + 1$$

$$= 0.4$$

$$= 1$$

$$V_3 = u_3 - \frac{(u_3, V_1)}{\|V_1\|^2} V_1 - \frac{(u_3, V_2)}{\|V_2\|^2} V_2$$

$$\begin{aligned}
 &= (0, 0, 1) - \frac{1}{3} (1, 1, 1) - \frac{0.4}{0.68} (-0.6, 0.4, 0.4) \\
 &= (0, 0, 1) - 0.3 (1, 1, 1) - 0.5 (-0.6, 0.4, 0.4) \\
 &= (0, 0, 1) - (0.3, 0.3, 0.3) + (0.3, -0.2, -0.2) \\
 &= (0 - 0.3 + 0.3, 0 - 0.3 - 0.2, 0 - 0.3 - 0.2) \\
 &= (0, -0.5, -0.5)
 \end{aligned}$$

$$\|V_3\|^2 = (V_3, V_3)$$

$$= (0, -0.5, -0.5) (0, -0.5, -0.5)$$

$$= 0 + 0.25 + 0.25 = 0.5$$

$$= 0.5$$

$$\|V_3\| = \sqrt{0.5}$$

$$\|V_1\| = \sqrt{3} = 1.7$$

$$\|V_2\| = \sqrt{0.68} = 0.8$$

$$\|V_3\| = \sqrt{0.5} = 0.7$$

$$\text{orthogonal sets} = [V_1, V_2, V_3] = \begin{bmatrix} 1 & -0.6 & 0 \\ 1 & 0.4 & -0.5 \\ 1 & 0.4 & -0.5 \end{bmatrix}$$

$$\text{orthonormal basis } Q = \left[\frac{V_1}{\|V_1\|}, \frac{V_2}{\|V_2\|}, \frac{V_3}{\|V_3\|} \right]$$

$$Q = \begin{bmatrix} \frac{1}{1.7} & \frac{-0.6}{0.8} & \frac{0}{0.7} \\ \frac{1}{1.7} & \frac{0.4}{0.8} & \frac{-0.5}{0.7} \\ \frac{1}{1.7} & \frac{0.4}{0.8} & \frac{-0.5}{0.7} \end{bmatrix}$$

$$Q = \begin{bmatrix} 0.5 & -0.75 & 0 \\ 0.5 & 0.5 & -0.7 \\ 0.5 & 0.5 & -0.7 \end{bmatrix}$$

$$Q^T = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ -0.75 & 0.5 & 0.5 \\ 0 & -0.7 & -0.7 \end{bmatrix}$$

$$R = Q^T A$$

$$R = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ -0.75 & 0.5 & 0.5 \\ 0 & -0.7 & -0.7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5+0.5+0.5 & 0+0.5+0.5 & 0+0+0.5 \\ -0.75+0.5+0.5 & 0+0.5+0.5 & 0+0+0.5 \\ 0+(-0.7)+(-0.7) & 0-0.7-0.7 & 0+0-0.7 \end{bmatrix}$$

$$R = \begin{bmatrix} 1.5 & 1 & 0.5 \\ 0.25 & 1 & 0.5 \\ 1.4 & 1.4 & -0.7 \end{bmatrix} = \begin{bmatrix} 1.5 & 1 & 0.5 \\ 0.25 & 1 & 0.5 \\ 0 & 0 & -0.7 \end{bmatrix}$$

$$A = QR$$

$$A = \begin{bmatrix} 0.5 & -0.75 & 0 \\ 0.5 & 0.5 & -0.7 \\ 0.5 & 0.5 & -0.7 \end{bmatrix} \begin{bmatrix} 1.5 & 1 & 0.5 \\ 0 & 1 & 0.5 \\ 0 & 0 & -0.7 \end{bmatrix}$$

12/3/26. Least square approximation.

$$\text{let } A = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ \vdots \\ t_n \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \quad x = \begin{bmatrix} c \\ d \end{bmatrix}$$

$$y = cb + d$$

$$E = \|Ax - y\|^2$$

Solve

1. Find the least square solution, the least error vector of the linear system $x_1 - x_2 = 4$, $3x_1 + 2x_2 = 1$, $-2x_1 + 4x_2 = 3$.

Solve

$$x_1 - x_2 = 4$$

$$3x_1 + 2x_2 = 1$$

$$-2x_1 + 4x_2 = 3$$

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \quad y = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

To find $A^T A = A^T y$

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+9+4 & -1+6-8 \\ -1+6-8 & 1+4+16 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix}$$

$$A^T Y = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 + 3 + (-6) \\ -4 + 2 + 12 \end{bmatrix}$$

$$A^T Y = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$$

$$A^T A x = A^T Y$$

$$\begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$$

$$14x_1 - 3x_2 = 1 \rightarrow \textcircled{1}$$

$$-3x_1 + 21x_2 = 10 \rightarrow \textcircled{2}$$

$$\textcircled{1} \times 7 = 98x_1 - 21x_2 = 7$$

$$-3x_1 + 21x_2 = 10$$

$$\hline 95x_1 = 17$$

$$x_1 = \frac{17}{95}$$

$$x_1 = 0.17$$

$$x_1 = 0.17 \text{ sub in } \textcircled{1}$$

$$14(0.17) - 3x_2 = 1$$

$$2.38 - 3x_2 = 1$$

$$-3x_2 = 1 - 2.38$$

$$-3x_2 = -1.38$$

$$3x_2 = 1.38$$

$$x_2 = \frac{1.38}{3}$$

$$x_2 = 0.46$$

$$y = x_1 t + x_2$$

$$y = 0.17t + 0.46$$

$$E = \|Ax - y\|^2$$

$$Ax = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 0.17 \\ 0.46 \end{bmatrix} = \begin{bmatrix} 0.17 - 0.46 \\ 0.51 + 0.92 \\ -0.34 + 1.84 \end{bmatrix}$$

$$(Ax - y) = \begin{bmatrix} 0.17 - 0.46 \\ 0.51 + 0.92 \\ -0.34 + 1.84 \end{bmatrix} = \begin{bmatrix} -0.29 \\ 1.43 \\ 1.5 \end{bmatrix}$$

$$= \begin{bmatrix} -0.29 \\ 1.43 \\ 1.5 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -4.29 \\ 0.43 \\ -1.5 \end{bmatrix}$$

$$\text{I.e. } \|Ax - y\|^2 = (-4.29, 0.43, -1.5)$$

$$(-4.29, 0.43, -1.5)$$

$$= 18.40 + 0.18 + 2.25$$

$$= 20.83$$

$$E = 20.83$$

$$y = 0.17t + 0.46$$

2. Find the least square solution of $Ax = b$, where

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

sol

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

To find $A^T A = A^T b$

$$A^T A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4+1+0 & 0+1+0 \\ 0-1+0 & 0+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2+0+0 \\ 0+0-2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$A^T A x = A^T b$$

$$\begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$5x_1 - x_2 = 2 \rightarrow \textcircled{1}$$

$$-x_1 + 5x_2 = -2 \rightarrow \textcircled{2}$$

$$\textcircled{1} \times 5 = 25x_1 - 5x_2 = 10$$

$$-x_1 + 5x_2 = -2$$

$$\hline 24x_1 = 8$$

$$x_1 = 8/24$$

$$x_1 = 1/3$$

Sub x_1 in $\textcircled{2}$

$$-1/3 + 5x_2 = -2$$

$$5(1/3) - x_2 = 2$$

$$5/3 - x_2 = 2$$

$$-x_2 = 2 - 5/3$$

$$-x_2 = 1/3$$

$$x_2 = -1/3$$

$$y = x_1 t + x_2$$

$$y = 1/3 t + (-1/3)$$

$$E = \|Ax - b\|^2$$

$$Ax = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix} = \begin{bmatrix} 2/3 + 0 \\ -1/3 - 1/3 \\ 0 - 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 \\ -2/3 \\ -2/3 \end{bmatrix}$$

$$Ax - b = \begin{bmatrix} 2/3 \\ -2/3 \\ -2/3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

$$\|Ax - b\|^2 = (-1/3, -2/3, 1/3) (-1/3, -2/3, 1/3)$$

$$= 1/9 + 4/9 + 1/9$$

$$= 6/9 = 2/3$$

$$E = 2/3$$

3. Find the least square solution.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

sol

To find $A^T A = A^T b$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+1 & 0+1+2 & 1-1-3 \\ 0+1+2 & 0+1+4 & 0-1-6 \\ 1-1-3 & 0-1-6 & 1+1+9 \end{bmatrix} = \begin{bmatrix} 3 & 3 & -3 \\ 3 & 5 & -7 \\ -3 & -7 & 11 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6+0+0 \\ 0+0+0 \\ 6+0+0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix}$$

$$A^T A x = A^T b$$

$$\begin{bmatrix} 3 & 3 & -3 \\ 3 & 5 & -7 \\ -3 & -7 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix}$$

$$3x_1 + 3x_2 - 3x_3 = 6 \rightarrow \textcircled{1}$$

$$3x_1 + 5x_2 - 7x_3 = 0 \rightarrow \textcircled{2}$$

$$-3x_1 - 7x_2 + 11x_3 = 6 \rightarrow \textcircled{3}$$

$$\sim \left[\begin{array}{ccc|c} 3 & 3 & -3 & 6 \\ 3 & 5 & -7 & 0 \\ -3 & -7 & 11 & 6 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 3 & 3 & -3 & 6 \\ 0 & 2 & -4 & -6 \\ 0 & -4 & 8 & 0 \end{array} \right] \begin{array}{l} R_2 = R_2 - R_1 \\ R_3 = R_3 + R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 3 & 3 & -3 & 6 \\ 0 & 4 & -8 & -12 \\ 0 & -4 & 8 & 12 \end{array} \right] R_2 = 2R_2$$

$$\sim \left[\begin{array}{ccc|c} 3 & 3 & -3 & 6 \\ 0 & 4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{array} \right] R_3 = R_3 + 2R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_1 + x_2 - x_3 = 2 \rightarrow \textcircled{1}$$

$$x_2 - 2x_3 = -3 \rightarrow \textcircled{2}$$

$$x_2 = 2x_3 - 3$$

Sub x_2 in $\textcircled{1}$.

$$x_1 + 2x_3 - 3 - x_3 = 2$$

$$x_1 + x_3 - 3 = 2$$

$$x_1 = 2 + 3 - x_3$$

$$x_1 = -x_3 + 5$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 + 5 \\ 2x_3 - 3 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 2x_3 \\ x_3 \end{bmatrix} + \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix}$$

$$x_3 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix}$$

$$x_3 = 1$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$$

2)

we have many least square solution

$$\begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

2) Find the least square solution of the system of equation
 $x_1 + x_2 + 3x_3 = 1$, $x_1 + x_2 + 3x_3 = 2$.

Sol

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

To find $A^T A = A^T b$.

$$A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1+1 & 1+1 & 3+3 \\ 1+1 & 1+1 & 3+3 \\ 3+3 & 3+3 & 9+9 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 2 & 2 & 6 \\ 2 & 2 & 6 \\ 6 & 6 & 18 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+2 \\ 1+2 \\ 3+6 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix}$$

$$A^T A x = A^T b$$

$$\begin{bmatrix} 2 & 2 & 6 \\ 2 & 2 & 6 \\ 6 & 6 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix}$$

$$2x_1 + 2x_2 + 6x_3 = 3$$

$$2x_1 + 2x_2 + 6x_3 = 3$$

$$6x_1 + 6x_2 + 18x_3 = 9$$

$$\sim \left[\begin{array}{ccc|c} 2 & 2 & 6 & 3 \\ 2 & 2 & 6 & 3 \\ 6 & 6 & 18 & 9 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & 2 & 6 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} R_2 = R_2 - R_1 \\ R_3 = R_3 - 3R_1 \\ \vdots \end{array}$$

$$\sim 2x_1 + 2x_2 + 6x_3 = 3 \rightarrow \textcircled{1}$$

$$\text{Sub. } x_2 = 1, x_3 = 1.$$

$$2x_1 + 2 + 6 = 3$$

$$2x_1 = 3 - 8$$

$$2x_1 = -5$$

$$x_1 = -5/2$$

$$\text{let } x_2 = 0, x_3 = 0.$$

$$2x_1 + 0 + 0 = 3$$

$$x_1 = 3/2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 3/2 \\ 0 \\ 0 \end{bmatrix}$$

- 5) For the set of data values use the least square approximation to find the fit and also find the errors for $\{(1,2) (3,4) (5,7) (7,9) (9,12)\}$.

Sol

$$y = ct + d$$

$$E = \|Ax - b\|^2$$

$$A = \begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 5 & 1 \\ 7 & 1 \\ 9 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 4 \\ 7 \\ 9 \\ 12 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A^T A = A^T b$$

$$A^T A = \begin{bmatrix} 1 & 3 & 5 & 7 & 9 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{bmatrix} = \begin{bmatrix} 1+9+25+49+81 & 1+3+5+7+9 \\ 1+3+5+7+9 & 1+1+1+1+1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 165 & 25 \\ 25 & 5 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 3 & 5 & 7 & 9 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 7 \\ 9 \\ 12 \end{bmatrix} = \begin{bmatrix} 2+12+35+63+108 \\ 2+4+7+9+12 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 220 \\ 34 \end{bmatrix}$$

$$A^T A x = A^T b$$

$$\begin{bmatrix} 165 & 25 \\ 25 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 220 \\ 34 \end{bmatrix}$$

$$165x_1 + 25x_2 = 220 \rightarrow \textcircled{1}$$

$$25x_1 + 5x_2 = 34 \rightarrow \textcircled{2}$$

$$165x_1 + 25x_2 = 220$$

$$5 \times \textcircled{2} \quad 125x_1 + 25x_2 = 170$$

$$40x_1 = 50$$

$$x_1 = 5/4$$

$$x_1 = 1.25$$

Sub in $\textcircled{1}$.

$$165 \cdot (1.25) + 25x_2 = 220$$

$$206.25 + 25x_2 = 220$$

$$25x_2 = 220 - 206.25$$

$$25x_2 = 13.75 \Rightarrow x_2 = 0.55$$

$$y = x_1 t + x_2$$

$$y = 1.25t + 0.55$$

$$E = \|Ax - b\|^2$$

$$Ax = \begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 5 & 1 \\ 7 & 1 \\ 9 & 1 \end{bmatrix} \begin{bmatrix} 1.25 \\ 0.55 \end{bmatrix} = \begin{bmatrix} 1.25 + 0.55 \\ 3.75 + 0.55 \\ 6.25 + 0.55 \\ 8.75 + 0.55 \\ 10.25 + 0.55 \end{bmatrix} = \begin{bmatrix} 1.8 \\ 4.3 \\ 6.8 \\ 9.3 \\ 11.8 \end{bmatrix}$$

$$Ax - b = \begin{bmatrix} 1.8 \\ 4.3 \\ 6.8 \\ 9.3 \\ 11.8 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 7 \\ 9 \\ 12 \end{bmatrix} = \begin{bmatrix} -0.2 \\ 0.3 \\ -0.2 \\ 0.3 \\ -0.2 \end{bmatrix}$$

$$\|Ax - b\|^2 = (-0.2, 0.3, -0.2, 0.3, -0.2)$$

$$\begin{aligned} & (-0.2, 0.3, -0.2, 0.3, -0.2) \\ & = 0.04 + 0.09 + 0.04 + 0.09 + 0.04 \\ & = 0.3 \end{aligned}$$

6. For the set of data values, use the least square approximation to find the fit and also find the errors for $\{(1,2), (2,3), (3,5), (4,7)\}$,
sok

$$y = ct + d$$

$$E = \|Ax - b\|^2$$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A^T A = A^T b.$$

$$A^T A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1+4+9+16 & 1+2+3+4 \\ 1+2+3+4 & 1+1+1+1 \end{bmatrix} = \begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 2+6+15+28 \\ 2+3+5+7 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 51 \\ 17 \end{bmatrix}$$

$$A^T A x = A^T b$$

$$\begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 51 \\ 17 \end{bmatrix}$$

$$30x_1 + 10x_2 = 51 \rightarrow \textcircled{1}$$

$$10x_1 + 4x_2 = 17 \rightarrow \textcircled{2}$$

$$\textcircled{1} \Rightarrow 30x_1 + 10x_2 = 51$$

$$3 \times \textcircled{2} \Rightarrow 30x_1 + 12x_2 = 51$$

$$2x_2 = 0$$

$$\boxed{x_2 = 0}$$

Sub x_2 in ①.

$$30x_1 + 10x_2 = 51$$

$$30x_1 + 10(0) = 51$$

$$30x_1 = 51$$

$$x_1 = 51/30$$

$$x_1 = 1.7$$

$$y = x_1 + x_2$$

$$y = 1.7 + 0$$

$$E = \|Ax - b\|^2$$

$$Ax = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1.7 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.7 + 0 \\ 3.4 + 0 \\ 5.1 + 0 \\ 6.8 + 0 \end{bmatrix}$$

$$Ax = \begin{bmatrix} 1.7 \\ 3.4 \\ 5.1 \\ 6.8 \end{bmatrix}$$

$$Ax - \overset{b}{y} = \begin{bmatrix} 1.7 \\ 3.4 \\ 5.1 \\ 6.8 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} -0.3 \\ 0.4 \\ 0.1 \\ -0.2 \end{bmatrix}$$

$$\|Ax - b\|^2 = (-0.3, 0.4, 0.1, -0.2)$$

$$(-0.3, 0.4, 0.1, -0.2)$$

$$= 0.09 + 0.16 + 0.01 + 0.04$$

$$= 0.30$$

$$\|Ax - b\|^2 = 0.30$$