UNIT – I LOGIC AND PROOFS

PART – A

1. **Get the contra positive of the statement "If it is raining then I get wet"**

Ans: Let p: it is raining and q: I get wet

Given $p \to q$. Its contra positive is given by $\neg q \to \neg p$

That is "If I don't get wet then it is not raining"

2. **Is it true that the negation of a conditional statement is also a conditional statement?**

Ans: No, because $\neg(p \to q) \equiv \neg(\neg p \vee q) \equiv p \wedge \neg q$

3. **Find a counter example,if possible, to these universally quantified statements, whose the universe of discourse for all variables consists of all integers.[November 2014]**

(a) $\forall x \forall y (x^2 = y^2 \to x = y)$.

(b) $\forall x \forall y (xy \geq x)$.

Ans: (a) $x = 2, y = -2$ and (b) $x = 17, y = -1$

4. **Show that the propositions $p \to q$ and $\neg p \vee q$ are logically equivalent.**

Ans:

P	q	$\neg p$	$\neg p \vee q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

5. **Show that $p \to (q \to r) \Leftrightarrow (p \wedge q) \to r$ without using truth tables.**

Ans: $p \to (q \to r) \Leftrightarrow \neg p \vee (\neg q \vee r) \Leftrightarrow (\neg p \vee \neg q) \vee r \Leftrightarrow \neg(p \wedge q) \vee r \Leftrightarrow (p \wedge q) \to r$

6. **Show that $(\neg p) \to (p \to q)$ is a tautology.**

Ans: $(\neg p) \to (p \to q) \Leftrightarrow p \vee (\neg p \vee q) \Leftrightarrow (p \vee \neg p) \vee q \Leftrightarrow T \vee q \Leftrightarrow T$

7. **Write the truth table for the formula $(p \wedge q) \vee (\neg p \wedge \neg q)$** **[November 2012]**

Ans:

P	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg p \wedge \neg q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$
T	T	F	F	T	F	T
T	F	F	T	F	F	F
F	T	T	F	F	F	F
F	F	T	T	F	T	T

8. **What are the negation of the statements $\forall x (x^2 > x)$ and $\exists x (x^2 = 2)$?** **[November 2013]**

Ans:

The negation of $\forall x (x^2 > x)$ is $\neg \forall x (x^2 > x)$

$\Leftrightarrow \exists x \neg (x^2 > x)$

$\Leftrightarrow \exists x (x^2 \leq x)$

The negation of $\exists x (x^2 = 2)$ is $\neg \exists x (x^2 = 2)$

$\Leftrightarrow \forall x \neg (x^2 = 2)$

$\Leftrightarrow \forall x (x^2 \neq 2)$

9.	**Express in symbolic form, everyone who is healthy can do all kinds of work. [November 2012]** **Ans:** Let $P(x)$: x is healthy and $Q(x)$: x do all work Symbolic form $\forall x(P(x) \to Q(x))$
10.	**Write the negation of the statement "If there is a will, then there is a way".** **Ans:** Let p: 'There is a will' and q: 'There is a way' Given $p \to q \Leftrightarrow \neg p \vee q$. Its negation is given by $p \wedge \neg q$ So, the negation of the given statement is "There is a will and there is no way"
11.	**When do you say that two compound propositions are equivalent?** **Ans:** Two statement formulas A and B are equivalent iff $A \leftrightarrow B$ or A □ B is a tautology. It is denoted by the symbol $A \Leftrightarrow B$ which is read as "A is equivalence to B"
12.	**Prove that $(p \leftrightarrow q) \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$ [November 2010]** **Ans:** $(p \leftrightarrow q) \Leftrightarrow (p \to q) \wedge (q \to p) \Leftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$ $\Leftrightarrow (\neg p \wedge \neg q) \vee (\neg p \wedge p) \vee (q \wedge \neg q) \vee (p \wedge q)$ $\Leftrightarrow (\neg p \wedge \neg q) \vee (p \wedge q)$
13.	**Rewrite the following using quantifiers "Every student in the class studied calculus".** **Ans:** Let $P(x)$: x is a student and $Q(x)$: x studied calculus Symbolic form $\forall x(P(x) \to Q(x))$
14.	**Check whether $((p \to q) \to r) \vee \neg p$ is a tautology.** **Ans:** $((p \to q) \to r) \vee \neg p \Leftrightarrow ((\neg p \vee q) \to r) \vee \neg p \Leftrightarrow (\neg(\neg p \vee q) \vee r) \vee \neg p \Leftrightarrow (p \wedge \neg q) \vee (r \vee \neg p)$ $\Leftrightarrow (r \vee \neg p \vee p) \wedge (r \vee \neg p \vee \neg q) \Leftrightarrow T \wedge (r \vee \neg p \vee \neg q) \Leftrightarrow (r \vee \neg p \vee \neg q)$ The given statement is not a tautology
15.	**Write the statement in symbolic form "Some real numbers are rational".** **Ans:** Let $R(x)$: x is a real number and $Q(x)$: x is rational Symbolic form: $\exists x (R(x) \wedge Q(x))$.
16.	**Show that $(p \to q) \wedge (q \to r)$ and $(p \vee q) \to r$ are logically equivalent. [November 2014]** **Ans:**For $(p \to q) \wedge (q \to r)$ to be false, one of the two implications must be false, which happens exactly when r is false and at least one of p and q is true, but these are precisely the cases in which $p \vee q$ is true and r is false. Which is precisely when $(p \vee q) \to r$ is false. Since the two propositions are false in exactly the same situations they are logically equivalent.
17.	**Define Compound statement formula.** **Ans:** An expression consisting of simple statement functions (one or more variables) connected by logical Connectives are called a compound statement.
18.	**Write the statement in symbolic form "Some integers are not square of any integers".** **Ans:** Let $I(x)$: x is an integer and S(x): x is a square of any integer Symbolic form: $\exists x (I(x) \wedge \neg S(x))$.
19.	**Define Contradiction.** **Ans:** A propositional formula which is always false irrespective of the truth values of the individual variables is a contradiction.

20.	**Define Universal quantification and Existential quantification.** **Ans:** The Universal quantification of a predicate formula $P(x)$ is the proposition, denoted by $\forall x P(x)$ that is true if P (a) is true for all subject a. The Existential quantification of a predicate formula $P(x)$ is the proposition, denoted by $\exists x P(x)$ that is true if P(a) is true for some subject a.
	PART – B
1(a)	**What is meant by Tautology? Without using truth table, show that** $((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$ **is a tautology.** **Solution**: A Statement formula which is true always irrespective of the truth values of the individual variables is called a tautology. Consider $\neg(\neg P \wedge (\neg Q \vee \neg R) \Rightarrow \neg(\neg P \wedge \neg(Q \wedge R) \Rightarrow P \vee (Q \wedge R) \Rightarrow (P \vee Q) \wedge (P \vee R) \quad (1)$ Consider $(\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R) \Rightarrow \neg(P \vee Q) \vee \neg(P \vee R) \Rightarrow \neg((P \vee Q) \wedge (P \vee R)) \quad (2)$ Using (1) and (2) $((P \vee Q) \wedge (P \vee Q) \wedge (P \vee R)) \vee \neg((P \vee Q) \wedge (P \vee R))$ $\Rightarrow [(P \vee Q) \wedge (P \vee R)] \vee \neg[(P \vee Q) \wedge (P \vee R)] \Rightarrow T$
1(b)	**Prove the following equivalences by proving the equivalences of the dual** $\neg((\neg P \wedge Q) \vee (\neg P \wedge \neg Q)) \vee (P \wedge Q) \equiv P$ **Solution:** It's dual is $\neg((\neg P \vee Q) \wedge (\neg P \vee \neg Q)) \wedge (P \vee Q) \equiv P$ Consider,

$\neg((\neg P \vee Q) \wedge (\neg P \vee \neg Q)) \wedge (P \vee Q) \equiv P$	**Reasons**
$\Rightarrow ((P \wedge \neg Q) \vee (P \wedge Q)) \wedge (P \vee Q)$	(Demorgan's law)
$\Rightarrow ((Q \wedge P) \vee (\neg Q \wedge P)) \wedge (P \vee Q)$	(Commutative law)
$\Rightarrow ((Q \vee \neg Q) \wedge P) \wedge (P \vee Q)$	(Distributive law)
$\Rightarrow (T \wedge P) \wedge (P \vee Q)$	$(P \vee \neg P \Rightarrow T)$
$\Rightarrow P \wedge (P \vee Q)$	$(P \wedge T = P)$
$\Rightarrow P$	(Absorption law)

2(a)	**Prove that** $(P \rightarrow Q) \wedge (R \rightarrow Q) \Leftrightarrow (P \vee R) \rightarrow Q$ **.** **Solution:**

$(P \rightarrow Q) \wedge (R \rightarrow Q)$	**Reasons**
$\Leftrightarrow (\neg P \vee Q) \wedge (\neg R \vee Q)$	Since $P \rightarrow Q \Leftrightarrow \neg P \vee Q$
$\Leftrightarrow (\neg P \wedge \neg R) \vee Q)$	Distribution law
$\Leftrightarrow \neg(P \vee R) \vee Q$	Demorgan's law
$\Leftrightarrow P \vee R \rightarrow Q$	since $P \rightarrow Q \Leftrightarrow \neg P \vee Q$

2(b)	**Obtain DNF of $Q \vee (P \wedge R) \wedge \neg((P \vee R) \wedge Q)$.** **Solution:** $Q \vee (P \wedge R) \wedge \neg((P \vee R) \wedge Q)$ $\Leftrightarrow (Q \vee (P \wedge R)) \wedge (\neg((P \vee R) \wedge Q)$ (Demorgan law) $\Leftrightarrow (Q \vee (P \wedge R)) \wedge ((\neg P \wedge \neg R) \vee \neg Q)$ (Demorgan law) $\Leftrightarrow (Q \wedge (\neg P \wedge \neg R)) \vee (Q \wedge \neg Q) \vee ((P \wedge R) \wedge \neg P \wedge \neg R) \vee ((P \wedge R) \wedge \neg Q)$ (Extended distributed law) $\Leftrightarrow (\neg P \wedge Q \wedge \neg R) \vee F \vee (F \wedge R \wedge \neg R) \vee (P \wedge \neg Q \wedge R)$ (Negation law) $\Leftrightarrow (\neg P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R)$ (Negation law)
3(a)	**Obtain Pcnf and Pdnf of the formula $(\neg P \vee \neg Q) \to (P \leftrightarrow \neg Q)$** **Solution:** Let S = $(\neg P \vee \neg Q) \to (P \leftrightarrow \neg Q)$ (table below) PCNF: $P \vee Q$ and PDNF: $(P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$
3(b)	**Obtain PDNF of $P \to (P \wedge (Q \to P))$.** **Solution:** $P \to (P \wedge (Q \to P)) \Leftrightarrow \sim P \vee (P \wedge (\sim Q \vee P))$ $\Leftrightarrow \sim P \vee (P \wedge \sim Q) \vee (P \wedge P)$ $\Leftrightarrow (\sim P \wedge T) \vee (P \wedge \sim Q) \vee (P \wedge P)$ $\Leftrightarrow (\sim P \wedge (Q \vee \sim Q)) \vee (P \wedge \sim Q)) \vee (P \wedge (Q \vee \sim Q))$ $\Leftrightarrow (\sim P \wedge Q) \vee (\sim P \wedge \sim Q) \vee (P \wedge \sim Q) \vee (P \wedge Q) \vee (P \wedge \sim Q)$ $\Leftrightarrow (\sim P \wedge Q) \vee (\sim P \wedge \sim Q) \vee (P \wedge \sim Q) \vee (P \wedge Q)$
4(a)	**Without constructing the truth table obtain the product-of-sums canonical form of the formula $(\neg P \to R) \wedge (Q \leftrightarrow P)$. Hence find the sum-of products canonical form.** **Solution:** Let $S \Leftrightarrow (\neg P \to R) \wedge (Q \leftrightarrow P)$ $\Leftrightarrow (\neg(\neg P) \vee R) \wedge ((Q \to P) \wedge (P \to Q))$ $\Leftrightarrow (P \vee R) \wedge (\neg Q \vee P) \wedge (\neg P \vee Q)$ $\Leftrightarrow [(P \vee R) \vee F] \wedge [(\neg Q \vee P) \vee F] \wedge [(\neg P \vee Q) \vee F]$ $\Leftrightarrow [(P \vee R) \vee (Q \wedge \neg Q) \wedge [(\neg Q \vee P) \vee (R \wedge \neg R)] \wedge [(\neg P \vee Q) \vee (R \wedge \neg R)]$ $\Leftrightarrow (P \vee R \vee Q) \wedge (P \vee R \vee \neg Q) \wedge (\neg Q \vee P \vee R) \wedge (\neg Q \vee P \vee \neg R) \wedge$ $(\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$ $S \Leftrightarrow (P \vee R \vee Q) \wedge (P \vee R \vee \neg Q) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$ (Pcnf)

Table for 3(a):

P	Q	$\neg$ P	$\neg$ Q	$\neg P \vee \neg$ Q	$P \leftrightarrow \neg Q$	S	Minterm	Maxterm
T	T	F	F	F	F	T	$P \wedge Q$	
T	F	F	T	T	T	T	$P \wedge \neg Q$	
F	T	T	F	T	T	T	$\neg P \wedge Q$	
F	F	T	T	T	F	F		$P \vee Q$

$\neg S \Leftrightarrow$ The remaining maxterms of P,Q and R.

$\therefore \neg S \Leftrightarrow (P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)$.

$\neg\neg(S) \Leftrightarrow$ Apply duality principle to $\neg S$

$S \Leftrightarrow (\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R)$ (PDNF)

4(b) **Obtain the PDNF and PCNF of** $P \vee (\neg P \rightarrow (Q \vee (\neg Q \rightarrow R)))$.

Solution:

$P \vee (\neg P \rightarrow (Q \vee (\neg Q \rightarrow R)))$

$\Rightarrow P \vee (P \vee (Q \vee (Q \vee R)))$

$\Rightarrow (P \vee Q \vee R)$

$S = (P \vee Q \vee R)$

$\neg S = (\neg P \vee Q \vee R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)$

$\wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee \neg R) \wedge (P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee R)$

$\neg\neg S = \neg((\neg P \vee Q \vee R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)$

$\wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee \neg R) \wedge (P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee R))$

$= (P \wedge \neg Q \wedge \neg R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R)$

$\vee (\neg P \wedge Q \wedge R) \vee (P \wedge \neg Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R)$

5(a) **Using indirect method of proof, derive p → ~s from the premises p → (q∨ r), q→ ~p, s→ ~r and p.**

Solution:

Let ~ (p → ~s) be an additional premise.

~(p → ~s) ⇔ ~(~p ∨ ~s) ⇔ (p ∧s)

1) p → (q∨ r)	Rule P
2) p	Rule P
3) (q∨ r)	Rule T, 1,2
4) p ∧s	Rule AP
5) s	Rule T,4
6) s→ ~r	Rule P
7) ~r	Rule T, 5, 6
8) q	Rule T3,7
9) q→ ~p	Rule P
10) ~P	Rule T, 8, 9
11) p ∧ ~p	Rule T, 2,10
12) F	Rule T, 11

5(b) **Prove that the premises** $a \rightarrow (b \rightarrow c), d \rightarrow (b \wedge \neg c)$, and $(a \wedge d)$ **are inconsistent.**

Solution:

{1}	$a \wedge d$	Rule P
{1}	a, d	Rule T
{3}	$a \rightarrow (b \rightarrow c)$	Rule P
{1,3}	$b \rightarrow c$	Rule T

{1,3}	$\neg b \vee c$	Rule T
{6}	$d \rightarrow (b \wedge \neg c)$	Rule P
{6}	$\neg(b \wedge \neg c) \rightarrow \neg d$	Rule T
{6}	$(\neg b \vee c) \rightarrow \neg d$	Rule T
{1,3,6}	$\neg d$	Rule T
{1,3,6}	$d \wedge \neg d$	Rule T

This is a false value. Hence the set of a premises are inconsistent

6(a) **Use the indirect method to prove that the conclusion $\exists z Q(z)$ follows from the premises $\forall x(P(x) \rightarrow Q(x))$ and $\exists y P(y)$**

Solution:

1	$\neg \exists z Q(z)$	P(assumed)
2	$\forall z \neg Q(z)$	T, (1)
3	$\exists y P(y)$	P
4	$P(a)$	ES, (3)
5	$\neg Q(a)$	US, (2)
6	$P(a) \wedge \neg Q(a)$	T, (4),(5)
7	$\neg(P(a) \rightarrow Q(a))$	T, (6)
8	$\forall x(P(x) \rightarrow Q(x))$	P
9	$P(a) \rightarrow Q(a)$	US, (8)
10	$P(a) \rightarrow Q(a) \wedge \neg(P(a) \rightarrow Q(a))$	T,(7),(9) contradiction

Hence the proof.

6(b) **Show that $R \rightarrow S$ can be derived from the premises $P \rightarrow (Q \rightarrow S)$, $\neg R \vee P$ & Q**

Solution:

R	Assumed premises
$\neg R \vee P$	Rule P
$R \rightarrow P$	Rule T
P	Rule T
$P \rightarrow (Q \rightarrow S)$	Rule P
$Q \rightarrow S$	Rule P
Q	Rule P
S	Rule T
$R \rightarrow S$	Rule CP

7(a) **Prove that $(x)(P(x) \rightarrow Q(x)), (x)(R(x) \rightarrow \neg Q(x)) \Rightarrow (x)(R(x) \rightarrow \neg P(x))$.**

Solution:

Step	**Derivation**	**Rule**
1	$(\forall x)(P(x) \rightarrow Q(x))$	P
2	$(\forall x)(R(x) \rightarrow \neg Q(x))$	P
3	$R(x) \rightarrow \neg Q(x)$	US, (2)
4	$R(x)$	P (assumed)
5	$\neg Q(x)$	T,(3),(4)

	6 $P(x) \to Q(x)$ US, (1) 7 $\neg P(x)$ T, (5),(6) 8 $R(x) \to \neg P(x)$ CP, (4),(7) 9 $(\forall x)(R(x) \to \neg P(x))$ UG, (9) Hence the argument is valid
7(b)	**Show that $(\exists x)(P(x) \wedge Q(x)) \Rightarrow (\exists x) P(x) \wedge (\exists x) Q(x)$** **Solution:** 1) $(\exists x)(P(x) \wedge Q(x))$ — Rule P 2) $P(a) \wedge Q(a)$ — ES, 1 3) $P(a)$ — Rule T, 2 4) $Q(a)$ — Rule T, 2 5) $(\exists x) P(x)$ — EG, 3 6) $(\exists x) Q(x)$ — EG, 4 7) $(\exists x) P(x) \wedge (\exists x) Q(x)$ — Rule T, 5, 6
8(a)	**Show that the following statements constitute a valid argument.** **If there was rain, then traveling was difficult. If they had umbrella, then traveling was not difficult.** **They had umbrella. Therefore there was no rain.** **Solution:** Let P : There was rain Q : Traveling was difficult R : They had umbrella Then, the given statements are symbolized as (1) $P \to Q$ (2) $R \to \sim Q$ (3) R Conclusion : ~P 1) R — Rule P 2) $R \to \sim Q$ — Rule P 3) ~ Q — Rule T,1,2 4) $P \to Q$ — Rule P 5) ~ P — Rule T,3,4 Therefore, it is a valid conclusion.
8(b)	**Show that the following premises are inconsistent.** **(1) If Nirmala misses many classes through illness then he fails high school.** **(2) If Nirmala fails high school, then he is uneducated.** **(3) If Nirmala reads a lot of books then he is not uneducated.** **(4) Nirmala misses many classes through illness and reads a lot of books.**

	Solution: E : Nirmala misses many classes S: Nirmala fails high school A: Nirmala reads lot of books H: Nirmala is uneducated Statement: (1) $E \rightarrow S$ (2) $S \rightarrow H$ (3) $A \rightarrow \sim H$ (4) $E \wedge A$ Premises are : $E \rightarrow S$, $S \rightarrow H$, $A \rightarrow \sim H$, $E \wedge A$ 1) $E \rightarrow S$ — Rule P 2) $S \rightarrow H$ — Rule P 3) E $\rightarrow$ H — Rule T, 1,2 4) $A \rightarrow \sim H$ — Rule P 5) H$\rightarrow$ ~A — Rule T,4 6) E$\rightarrow$ ~A — Rule T,3,5 7) $\sim E \vee \sim A$ — Rule T,6 8) ~(E$\wedge$ A) — Rule T,7 9) E$\wedge$ A — Rule P 10) (E$\wedge$ A) $\wedge$ ~ (E$\wedge$ A) — Rule T,8,9 Which is nothing but false Therefore given set of premises are inconsistent
9(a)	**Show that the hypotheses,"It is not sunny this afternoon and it is colder than yesterday," " We will go swimming only if it is sunny," "If we do not go swimming then we will take a canoe trip," and "If we take a canoe trip, then we will be home by sunset "lead to the conclusion "we will be home by sunset".** **Solution:** p – It is sunny this afternoon. q- It is colder than yesterday r- we will go swimming s- we will take a canoe trip t- we will be home by sunset The given premises are $\neg p \wedge q, r \rightarrow p, \neg r \rightarrow s$ & $s \rightarrow t$ Step — Reason $\neg p \wedge q$ — Hypothesis $\neg p$ — step 1 $r \rightarrow p$ — Hypothesis $\neg r$ — moduus tollens step 2 &3

$\neg r \to s$	Hypothesis
s	modus ponens step 4 &5
$s \to t$	Hypothesis
t	modus ponens step 6&7

9(b) **Prove that $\sqrt{2}$ is irrational by giving a proof using contradiction.**

Solution:

Let P: $\sqrt{2}$ is irrational.

Assume ~P is true, then $\sqrt{2}$ is rational, which leads to a contradiction.

By our assumption is $\sqrt{2} = \frac{a}{b}$, where a and b have no common factors ---------------(1)

$\Rightarrow 2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow a^2$ is even. $\Rightarrow a = 2c$

$2b^2 = 4c^2 \Rightarrow b^2 = 2c^2 \Rightarrow b^2$ is even $\Rightarrow b$ is even as well.

$\Rightarrow$ a and b have common factor 2 (since a and b are even)

But it contradicts (1)

This is a contradiction.

Hence ~P is false.

Thus P: $\sqrt{2}$ is irrational is true.

10(a) **Let p, q, r be the following statements:**

p: I will study discrete mathematics

q: I will watch T.V.

r: I am in a good mood.

Write the following statements in terms of p, q, r and logical connectives.

(1) If I do not study and I watch T.V., then I am in good mood.

(2) If I am in good mood, then I will study or I will watch T.V.

(3) If I am not in good mood, then I will not watch T.V. or I will study.

(4) I will watch T.V. and I will not study if and only if I am in good mood.

Solution:

(1) $(\neg p \wedge q) \to r$

(2) $r \to (p \vee q)$

(3) $\neg r \to (\neg q \vee p)$

(4) $(q \wedge \neg p) \leftrightarrow r$

10(b) **Give a direct proof of the statement."The square of an odd integer is an odd integer".**

Solution:

Given: The square of an odd integer is an odd integer".

P: n is an odd integer.

Q:n^2 is an odd integer

Hypothesis: Assume that P is true

Analysis : n=2k+1 where k is some integer.

$n^2=(2k+1)^2=2(2k^2+2k)+1$

Conclusion: n^2 is not divisible by 2.Therefore n^2 is an odd integer.

$P \to Q$ is true.

	UNIT II COMBINATORICS
	PART – A
1.	**State pigeon hole principle.** **Ans:** If (n+1) pigeons occupies n holes then at least one hole has more than 1 pigeon.
2.	**State the generalized pigeon hole principle.** **Ans:** If *m* pigeons occupies *n* holes (*m*>*n*), then at least one hole has more than $\left\lfloor \frac{m-1}{n} \right\rfloor + 1$ pigeons.
3.	**Show that, among 100 people, at least 9 of them were born in the same month.** **Ans:** Here no.of pigeon =m= no. of people =100 No. of holes = n= no. of month =12 Then by generalized pigeon hole principle, $\left\lfloor \frac{100-1}{12} \right\rfloor + 1 = 9$ were born in the same month.
4.	**In how many ways can 6 persons occupy 3 vacant seats?** **Ans:** Total no of ways $= 6c_3$ = 20 ways.
5.	**How many permutations of the letters in ABCDEFGH contain the string ABC .** **Ans:** Because the letters ABC must occur as block, we can find the answer by finding no of permutation of six objects, namely the block ABC and individual letters D,E,F,G and H . Therefore, there are 6! =720 permutations of the letters in ABCDEFGH which contains the string ABC.
6.	**How many different bit strings are there of length 7?** **Ans:** By product rule, 2^7=128 ways
7.	**How many ways are there to form a committee, if the committee consists of 3 educationalists and 4 socialist, if there are 9 educationalists and 11 socialist?** **Ans:** The 3 educationalist can be chosen from 9 educationalists in $9c_3$ ways. The 4 socialist can be chosen from 11 socialist in $11C_4$ ways. By product rule, the no of ways to select, the committee is = $9C_3.11C_4$ = 27720 ways.
8.	**There are 5 questions in a question paper in how many ways can a boy solve one or more questions?** **Ans:** The boy can dispose of each question in two ways .He may either solve it or leave it. Thus the no. of ways of disposing all the questions= 2^5 . But this includes the case in which he has left all the questions unsolved. The total no of ways of solving the paper = $2^5 - 1$ = 31.
9.	**If the sequence $a_n = 3.2^n, n \geq 1$, then find the corresponding recurrence relation.** **Ans:** For n ≥ 1 $a_n = 3.2^n,\ a_{n-1} = 3.2^{n-1} = 3.\frac{2^n}{2} \Rightarrow a_{n-1} = \frac{a_n}{2} \Rightarrow 2a_{n-1} = a_n$ $a_n = 2a_{n-1}$, for n≥1, with $a_0 = 3$.
10.	**If seven colours are used to paint 50 bicycles, then show that at least 8 bicycles will be the same colour.** **Ans:** Here, No. of Pigeon $= m =$ No. of bicycle=50 No. of Holes $= n =$ No. of colours = 7 By generalized pigeon hole principle, we have $\left\lfloor \frac{50-1}{7} \right\rfloor + 1 = 8$

11.	**Find the recurrence relation whose solution is $S(k) = 5.2^k$** **Ans:** Given $S(k) = 5.2^k \Rightarrow S(k-1) = 5.2^{k-1} = \frac{5}{2}.2^k \Rightarrow 2S(k-1) = 5.2^k = S(k)$ $2S(k-1) - S(k) = 0, \; with \; S(0) = 5$ is the required recurrence relation.
12.	**Find the associated homogeneous solution for $a_n = 3a_{n-1} + 2n$.** **Ans:** Its associated homogeneous equation is $a_n - 3a_{n-1} = 0$ Its characteristic equation is r-3 =0 $\Rightarrow$ r =3 Now, the solution of associated homogeneous equation is $a_n = A.3^n$
13.	**Solve $S(k) - 7S(k-1) + 10S(k-2) = 0$** **Ans:** The associated homogeneous relation is $S(k) - 7S(k-1) + 10S(k-2) = 0$ Its characteristic equation is $r^2 - 7r + 10 = 0 \Rightarrow (r-2)(r-5) = 0 \Rightarrow$ r =2,5 The solution of associated homogeneous equation is $S_k = A.2^k + B.5^k$
14.	**Define Generating function.** **Ans:** The generating function for the sequence‘s’ with terms $a_0, a_1, \ldots\ldots a_n \ldots$,of real numbers is the infinite sum . G(x) = G(s,x) = $a_0 + a_1 x + \ldots\ldots + a_n x^n + \ldots = \sum_{n=0}^{\infty} a_n x^n$.
15.	**Find the generating function for the sequence ‘s’ with terms 1,2,3,4……..** **Ans:** $G(x) = G(s,x) = \sum_{n=0}^{\infty} (n+1)x^n = 1 + 2x + 3x^2 + \ldots\ldots = (1-x)^{-2} = \frac{1}{(1-x)^2}$.
16.	**How many permutations of (a, b, c, d, e, f, g) end with a? [November 2014]** **Ans:** 6!$\times$ 1!=720
17.	**Find the number of arrangements of the letters in MAPPANASSRR.** **Ans:** Number of arrangements $= \frac{11!}{3!2!2!} = \frac{3991680}{48}$
18.	**In how many ways can letters of the word “INDIA” be arranged?** **Ans:** The word contains 5 letters of which 2 are I’s. The number of words possible $= \frac{5!}{2!} = 60$.
19.	**How many students must be in a class to guarantee that atleast two students receive the same score on the final exam if the exam is graded on a scale from 0 to 100 points.** **Ans:** There are 101 possible scores as 0, 1, 2, …,100. By Pigeon hole principle, we have among 102 students there must be atleast two students with the same score. The class should contain minimum 102 students.
20	**Show that among any group of five (not necessarily consecutive) integers, there are two with same remainder when divided by 4.** **Ans:** Take any group of five integers. When these are divided by 4 each have some remainder. Since there are five integers and four possible remainders when an integer is divided by 4, the pigeonhole principle implies that given five integers, atleast two have the same remainder.

	PART – B
1(a)	**Using Mathematical induction prove that** $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ **Solution:** Let P(n) : $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ (1) Assume P(1) : $1^2 = \frac{1(1+1)(2.1+1)}{6}$ is true (2) Assume P(k) : $1^2 + 2^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$ is true, where k is any integer. (3) P(k + 1) = $1^2 + 2^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$ $= \frac{(k+1)[(k+1)+1][(2(k+1)+1]}{6}$ Therefore P(k + 1) is true. Hence, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ is true for all n.
1(b)	**Use mathematical Induction to prove that $\left(3^n + 7^n - 2\right)$ is divisible by 8, for $n \geq 1$.** **Solution:** Let $P(n)$: $\left(3^n + 7^n - 2\right)$ is divisible by 8. (i) $P(1)$: $\left(3^1 + 7^1 - 2\right)$ 8 is divisible by 8, is true. (ii) Assume $P(k)$: $\left(3^k + 7^k - 2\right)$ is divisible by 8 is true ------(1) Claim: $P(k+1)$ is true $P(k+1) = 3^{k+1} + 7^{k+1} - 2$ $= 3\cdot 3^k + 7\cdot 7^k - 2$ $= 3.3^k + 3\cdot 7^k + 4\cdot 7^k - 6 + 4$ $= 3\left(3^k + 7^k - 2\right) + 4\left(7^k + 1\right)$ $\therefore$ $4\left(7^k + 1\right)$ is divisible by 8 and by (1) $3\left(3^k + 7^k - 2\right)$ is divisible by 8. $P(k+1) = 3\left(3^k + 7^k - 2\right) + 4\left(7^k + 1\right)$ is divisible by 8 is true.
2(a)	**Prove by mathematical induction that $6^{n+2} + 7^{2n+1}$ is divisible by 43 for each positive integer n.** **Solution:** S(1): Inductive step: for $n = 1$, $6^{1+2} + 7^{2+1}$=559, which is divisible by 43 So S(1) is true. Assume S(k) is true (i.e) $6^{k+2} + 7^{2k+1} = 43m$ for some integer m.

	To prove S(k+1) is true.Now $6^{k+3} + 7^{2k+3} = 6^{k+3} + 7^{2k+1}.7^2$ $= 6(6^{k+2} + 7^{2k+1}) + 43.7^{2k+1}$ $= 6.43m + 43.7^{2k+1}$ $= 43(6m + 7^{2k+1})$ Which is divisible by 43. So S(k+1) is true. By Mathematical Induction , S(*n*) is true for all integer n.
2(b)	**Using mathematical induction ,prove that** $2 + 2^2 + 2^3 + \dots\dots\dots + 2^n = 2^{n+1} - 2$ Let p (n) = $2 + 2^2 + 2^3 + \dots\dots\dots + 2^n$. Assume p (1): $2^1 = 2^{1+1} - 2$ is true. Assume p(k) : $2 + 2^2 + 2^3 + \dots\dots\dots + 2^k = 2^{k+1} - 2$ is true Claim p(k+1) is true. P(k+1) : $2 + 2^2 + 2^3 + \dots + 2^k + 2^{k+1} = 2^{k+1} - 2 + 2^{k+1} = 2.2^{k+1} - 2 = 2^{k+2} - 2$ P(k+1) is true. Hence it is true for all n.
3(a)	**Suppose there are six boys and five girls,** **(1) In how many ways can they sit in a row.** **(2) In how many ways can they sit in a row, if the boys and girls each sit together.** **(3) In how many ways can they sit in a row, if the girls are to sit together and the boy don't sit together.** **(4) How many seating arrangements are there with no two girls sitting together.** **Solution:** 1. There are 6 + 5 = 11 persons and they can sit in $11P_{11}$ ways. $11P_{11}$ = 11! ways 2. The boys among themselves can sit in 6! ways and girls among themselves can sit in 5! ways.They can be considered as 2 units and can be permuted in 2! ways. Thus the required seating arrangement can be done in = 2! x 6! x 5! ways = 172800 ways 3. The boys can sit in 6! Ways and girls in 5! ways. Since girls have to sit together they are considered as one unit. Among the 6 boys either 0 or 1 or 2 or 3 or 4 or 5 or 6 have to sit to the left of the girls units. Of these seven ways 0 and 6 cases have to be omitted as the boys do not sit together.Thus the required number of arrangements = 5 x 6! x 5! = 432000 ways. 4. The boys can sit in 6 ! ways. There are seven places where the girls can be placed. Thus the total arrangements are $7P_5$ x 6! Ways = 1814400 ways.
3(b)	**A bit is either 0 or 1 . A byte is a sequence of 8 bits. Find the number of bytes.Among these how many are (i) Starting with 11 and ending with 00 (ii) Starting with 11 but not ending with 00.** **Solution:** (1) Consider a byte starting with 11 and ending with 00.Now the remaining 4 places can be filled with either 0 or 1 which can be done in 2^4.Hence there are 16 bytes starting with 00 and ending with11. (2) Consider a byte starting with 11 and not ended with 00 Now there are 3 bytes which is not ended with 00(ended with 01,10 and 11).Now the remaining 4 places can be filled with either 0 or 1 which can be done in 2^4ways.Hence there are 3×16=48 bytes starting with 00 but not ending with11
4(a)	**How many positive integers '*n*' can be formed using the digits 3,4,4,5,5,6,7 if '*n*' has to exceed 50,00,000 ?** **Solution:** Consider a 7digit number $p_1, p_2, p_3, p_4, p_5, p_{6,} p_7$, in order to be a number ≥ 5000000 , p_1 is filled with

	either 5 or 6 or 7 (mutually exclusive) Case(1): p_1 is filled with 5 and remaining 6 position are filled with 3, 4, 4(repeated),5,6,7 in $= \frac{6!}{2!} = 360$ Case(2): p_1 is filled with 6 and remaining 6 positions are filled with 3,4,4 (repeated) 5,5 (repeated), 7 in $= \frac{6!}{2!2!} = 180$ Case(3) p_1 is filled with 7 and remaining 6 position are filled with 3,4,4(repeated),5,5 (repeated), 6 in $= \frac{6!}{2!2!} = 180$ All above 3 cases are mutually exclusive in total 360+180+180=720 ways.
4(b)	**Prove that in any group of six people there must be atleast three mutual friends or three mutual enemies.** **Proof:** Let the six people be A, B, C, D, E and F. Fix A. The remaining five people can accommodate into two groups namely (1) Friends of A and (2) Enemies of A Now by generalized Pigeon hole principle, at least one of the group must contain $\left(\frac{5-1}{2}\right)+1=3$ people. Let the friend of A contain 3 people.(Let it be B, C, D) Case(1) If any two of these three people (B, C, D) are friends, then these two together with A form three mutual friends. Case(2) If no two of these three people are friends, then these three people (B, C, D) are mutual enemies. In either case, we get the required conclusion. If the group of enemies of A contains three people, by the above similar argument, we get the required conclusion.
5(a)	**A computer password consists of a letter of English alphabet followed by 2 or 3 digits. Find the following :** **(1) The total number of passwords that can be formed** **(2) The number of passwords that no digit repeats.** **Sol**: (1) Since there are 26 alphabets and 10 digits and the digits can be repeated by the product rule the number of 3-character password is 26.10.10=2600 Similarly the number of 4 character password is 26.10.10.10=26000 Hence the tool number of password is 2600+26000=28600. (2) Since the digits are not repeated, the first digit after alphabet can be taken from any one out of 10, the second digit from remaining 9 digits and so on. Thus the number of 3-character password is 26.10.9=2340 Similarly the number of 4- character password is 26.10.9.8=18720 Hence the total number of password is 2340+18720=21060.
5(b)	**Show that among $(n+1)$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.** Solution: Let the $(n+1)$ integers be $a_1, a_2, \ldots a_{n+1}$ Each of these numbers can be expressed as an odd multiple of a power of 2. i.e $a_i = 2^{ki} \times m_i$ Where k_i non negative integer m_i odd integer where i =1, 2,3,..., n +1.

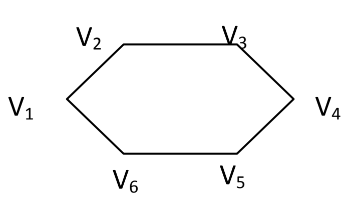
	Here, Pigeon=The odd positive integers $m_1, m_2, \ldots m_{n+1}$ less than 2_n Pigeon= 'n' odd positive integer less than $2n$. Hence by pigeon hole principle, 2 of the integers must be equal. Now $a_i = 2^{ki} m_i$ and $a_j = 2^{kj} m_j$ $\frac{a_i}{a_j} = \frac{2^{ki}}{2^{kj}} \quad (\because m_i = m_j)$ Case-1: If $k_i < k_j$ then 2^{k_i} divides 2^{k_j} and hence a_i divides a_j. Case-2: If $k_i > k_j$ then a_j divides a_i.
6(a)	**In A survey of 100 students, it was found that 30 studied Mathematics, 54 studied Statistics, 25 studied Operations Research, 1 studied all the three subjects, 20 studied Mathematics and Statistics, 3 studied Mathematics and Operation Research and 15 studied Statistics and Operation Research. Find how many students studied none of these subjects and how many students studied only Mathematics?** **Solution.** n(A) = 30; n(B) = 54;n(C) = 25; n(A∩B) = 20; n(A∩C) = 3; n(B∩C) = 15; n(A∩B∩C)=1 n(A∪B∪C) = n(A)+n(B)+n(C) – n(A∩B) – n(B∩C) – n(A∩C) + n(A∩B∩C) = 72 None of the subjects = 28. Only mathematics = 8.
6(b)	**A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and Russian, 23 have taken courses in both Spanish and French and 14 have taken courses in both French and Russian. If 2092 students have taken atleast one of Spanish, French and Russian, how many students have taken a course in all three languages?** **Solution:** S-Spanish,F-French, R-Russian |S|=1232 |F|=879 |R|=114 |S∩R|=103 |S∩F|=23 |F∩R|=14 |S∪F∪R|=2092 |S∪F∪R=|S|+|F|+|R|-|S∩F|- |S∩R|-|F∩R|+|S∩F∩R| ∴ |S∩F∩R|=7
7(a)	**Find all the solution of the recurrence relation $a_n = 5a_{n-1} - 6\, a_{n-2} + 7^n$** **Solution:** Given non-homogeneous equation can be written as $a_n - 5a_{n-1} + 6\, a_{n-2} - 7^n = 0$ Now, its associated homogeneous equation is $a_n - 5a_{n-1} + 6\, a_{n-2} = 0$ Its characteristic equation is $r^2 - 5r + 6 = 0$ Roots are r = 2,3 Solution is $a_n^{(h)} = c_1 2^n + c_2 3^n$ To find particular solution Since $F(n) = 7^n$, then the solution is of the form $C.7^n$, where C is a constant. Therefore, the equation $a_n = 5a_{n-1} - 6\, a_{n-2} + 7^n$ becomes $C7^n = 5C7^{n-1} - 6C7^{n-2} + 7^n$ ……(1) Dividing the both sides of (1) by 7^{n-2}. $(1) \to \frac{C.7^n}{7^{n-2}} = \frac{5C7^{n-1}}{7^{n-2}} - \frac{6C7^{n-2}}{7^{n-2}} + \frac{7^n}{7^{n-2}} \to C = \frac{49}{20}$

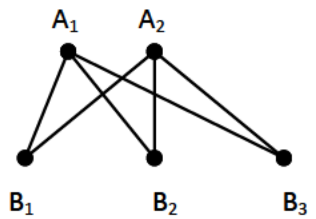
	Hence the particular solution is $a_n^{(p)} = \left(\frac{49}{20}\right)7^n$ Therefore, $a_n = c_1(2)^n + c_2(3)^n + \left(\frac{49}{20}\right)7^n$
7(b)	**Find the number of integers between 1 and 250 that are not divisible by any of the integers 2, 3, 5 &7.** Sol: Let A, B, C,D are the set of integers between 1 and 250 that are divisible by 2, 3, 5, 7 respectively. $\therefore \quad \lvert A \rvert = [\frac{250}{2}] = 125, \quad \lvert B \rvert = [\frac{250}{3}] = 83$ $\lvert C \rvert = [\frac{250}{5}] = 50, \quad \lvert D \rvert = [\frac{250}{7}] = 35$ $\lvert A \cap B \rvert = \left\lfloor \frac{250}{LCM\ (2,3)} \right\rfloor = \left\lfloor \frac{250}{2 \times 3} \right\rfloor = \left\lfloor \frac{250}{6} \right\rfloor = 41$ $\lvert A \cap C \rvert = \left\lfloor \frac{250}{LCM\ (2,5)} \right\rfloor = \left\lfloor \frac{250}{2 \times 5} \right\rfloor = \left\lfloor \frac{250}{10} \right\rfloor = 25$ $\lvert A \cap D \rvert = \left\lfloor \frac{250}{LCM\ (2,7)} \right\rfloor = \left\lfloor \frac{250}{2 \times 7} \right\rfloor = \left\lfloor \frac{250}{14} \right\rfloor = 17$ $\lvert B \cap C \rvert = \left\lfloor \frac{250}{LCM\ (3,5)} \right\rfloor = \left\lfloor \frac{250}{5 \times 3} \right\rfloor = \left\lfloor \frac{250}{15} \right\rfloor = 16$ $\lvert B \cap D \rvert = \left\lfloor \frac{250}{LCM\ (7,3)} \right\rfloor = \left\lfloor \frac{250}{7 \times 3} \right\rfloor = \left\lfloor \frac{250}{21} \right\rfloor = 11$ $\lvert C \cap D \rvert = \left\lfloor \frac{250}{LCM\ (5,7)} \right\rfloor = \left\lfloor \frac{250}{5 \times 7} \right\rfloor = \left\lfloor \frac{250}{35} \right\rfloor = 7$ $\lvert A \cap B \cap C \rvert = \left\lfloor \frac{250}{LCM\ (2,3,5)} \right\rfloor = \left\lfloor \frac{250}{2 \times 3 \times 5} \right\rfloor = 8$ $\lvert A \cap B \cap D \rvert = \left\lfloor \frac{250}{LCM\ (2,3,7)} \right\rfloor = \left\lfloor \frac{250}{2 \times 3 \times 7} \right\rfloor = 5$ $\lvert A \cap C \cap D \rvert = \left\lfloor \frac{250}{LCM\ (2,5,7)} \right\rfloor = \left\lfloor \frac{250}{2 \times 5 \times 7} \right\rfloor = 3$ $\lvert B \cap C \cap D \rvert = \left\lfloor \frac{250}{LCM\ (3,5,7)} \right\rfloor = \left\lfloor \frac{250}{3 \times 5 \times 7} \right\rfloor = 2$ $\lvert A \cap B \cap C \cap D \rvert = \left\lfloor \frac{250}{LCM\ (2,3,5,7)} \right\rfloor = \left\lfloor \frac{250}{2 \times 3 \times 5 \times 7} \right\rfloor = 1$ $\lvert A \cup B \cup C \cup D \rvert = \lvert A \rvert + \lvert B \rvert + \lvert C \rvert + \lvert D \rvert - \lvert A \cap B \rvert - \lvert A \cap C \rvert - \lvert A \cap D \rvert - \lvert B \cap C \rvert$ $- \lvert B \cap D \rvert + \lvert C \cap D \rvert + \lvert A \cap B \cap C \rvert + \lvert A \cap B \cap D \rvert + \lvert A \cap C \cap D \rvert$ $+ \lvert B \cap C \cap D \rvert - \lvert A \cap B \cap C \cap D \rvert$ =125+83+50+35-41-25-17-16-11-7+8+5+3+2-1=193 The number of integers between 1 and 250 that is divisible by any of the integers 2, 3, 5 and 7=193 Therefore not divisible by any of the integers 2, 3, 5 and 7=250-193=57.

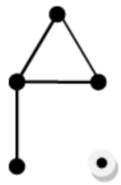
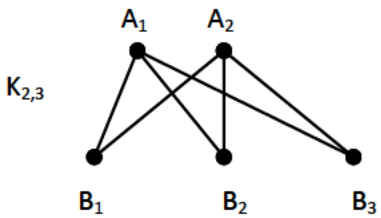
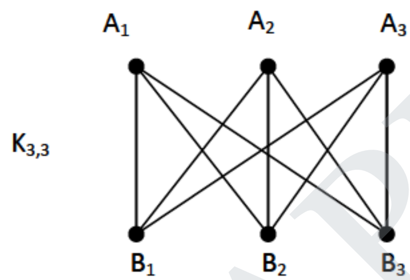
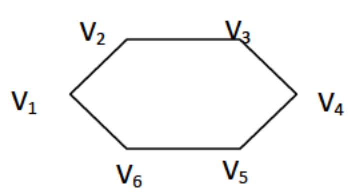
8(a)	**Solve the recurrence relation** $a_n = 2(a_{n-1} - a_{n-2})$ **where** $n \geq 2$ **and** $a_0 = 1, a_1 = 2$ $a_n = 2(a_{n-1} - a_{n-2})$ $= a_n - 2a_{n-1} + 2a_{n-2} = 0$ The characteristic equation is given by $\lambda^2 - 2\lambda + 2 = 0$ $\therefore \lambda = \dfrac{2 \pm \sqrt{4 - 4(2)}}{2} = \dfrac{2 \pm i2}{2} = 1 \pm i$ $\therefore \lambda = 1 + i, 1 - i$ $\therefore$ Solution is $a_n = A(1+i)^n + B(1-i)^n$ Where A and B are arbitrary constants Now, we have $z = x + iy$ $= r[\cos\theta + i\sin\theta]$ $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$ By Demoivre's theorem we have, $(1+i)^n = [\sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)]^n$ $= [\sqrt{2}]^n\left(\cos\dfrac{n\pi}{4} + i\sin\dfrac{n\pi}{4}\right)$ and $(1-i)^n = [\sqrt{2}]^n\left(\cos\dfrac{n\pi}{4} - i\sin\dfrac{n\pi}{4}\right)$ Now, $a_n = A[[\sqrt{2}]^n\left(\cos\dfrac{n\pi}{4} + i\sin\dfrac{n\pi}{4}\right)] + B[[\sqrt{2}]^n\left(\cos\dfrac{n\pi}{4} - i\sin\dfrac{n\pi}{4}\right)]$ $= [\sqrt{2}]^n\left((A+B)\cos\dfrac{n\pi}{4} + i(A-B)\sin\dfrac{n\pi}{4}\right)$ $\therefore a_n = [\sqrt{2}]^n\left(C_1\cos\dfrac{n\pi}{4} + C_2\sin\dfrac{n\pi}{4}\right)]$ (1) Is the required solution. Let $C_1 = A + B$, $C_2 = i(A - B)$ Since $a_0 = 1, a_1 = 2$

	$(1) \Rightarrow a_0 = (\sqrt{2})[C_1 \cos 0 + C_2 \sin 0] = 0$ $\Rightarrow 1 = C_1$ $a_1 = [\sqrt{2}]^1 \left(C_1 \cos \frac{\pi}{4} + C_2 \sin \frac{\pi}{4} \right)]$ $2 = \sqrt{2} \left(C_1 \frac{1}{\sqrt{2}} + C_2 \sin \frac{1}{\sqrt{2}} \right)]$ $\Rightarrow 2 = C_1 + C_2$ $\Rightarrow C_2 = 1$ $\therefore a_n = [\sqrt{2}]^n \left(\cos \frac{n\pi}{4} + \sin \frac{n\pi}{4} \right)]$
8(b)	**Solve the recurrence relation of the Fibonacci sequence of numbers** $f_n = f_{n-1} + f_{n-2}, n > 2$ **with initial conditions** $f_1 = 1, f_2 = 1$. Sol: The sequence of Fibonacci numbers satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ ..... (1) and satisfies the initial conditions $f_1 = 1, f_2 = 1$. $(1) \Rightarrow f_n - f_{n-1} - f_{n-2} = 0$ ...(2) Let $f_n = r^n$ be a solution of the given equation. The characteristic equation is $r^2 - r - 1 = 0$ $r = \frac{1 \pm \sqrt{1+4}}{2}$ Let $r_1 = \frac{1+\sqrt{5}}{2}, r_2 = \frac{1-\sqrt{5}}{2}$ $\therefore$ By theorem $f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right)^n$ ...(3) $f_1 = 1 \Rightarrow f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right) = 1$ $(1+\sqrt{5})\alpha_1 + (1-\sqrt{5})\alpha_2 = 2$ ...(4) $f_2 = 1 \Rightarrow f_2 = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right)^2 + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right)^2 = 1$ $= \alpha_1 \frac{(1+\sqrt{5})^2}{4} + \alpha_2 \frac{(1-\sqrt{5})^2}{4} = 1$ $= (1+\sqrt{5})^2 \alpha_1 + (1-\sqrt{5})^2 \alpha_2 = 4$ ...(5)

	$(4)\times(1-\sqrt{5})\Rightarrow$ $(1-\sqrt{5})\quad(1+\sqrt{5})\alpha_1+(1-\sqrt{5})^2\alpha_2=2\quad(1-\sqrt{5})\quad\ldots(6)$ $(6)-(5)\Rightarrow\ \alpha_1(1+\sqrt{5})[1-\sqrt{5}-1-\sqrt{5}]=2-2\sqrt{5}-4$ $\alpha_1(1+\sqrt{5})[-2\sqrt{5}]=-2-2\sqrt{5}$ $\alpha_1(1+\sqrt{5})[-2\sqrt{5}]=-2(1+\sqrt{5})$ $\alpha_1=\frac{1}{\sqrt{5}}$ $4)\Rightarrow\ (1+\sqrt{5})\frac{1}{\sqrt{5}}+(1-\sqrt{5})\alpha_2=2$ $\frac{1}{\sqrt{5}}+1+(1-\sqrt{5})\alpha_2=2$ $(1-\sqrt{5})\alpha_2=2-\frac{1}{\sqrt{5}}-1$ $=1-\frac{1}{\sqrt{5}}$ $(1-\sqrt{5})\alpha_2=\frac{\sqrt{5}-1}{\sqrt{5}}$ $\alpha_2=\frac{-1}{\sqrt{5}}$ $(3)\Rightarrow\ f_n=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n+\frac{-1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$
9(a)	**Solve the recurrence relation** $a_n=6a_{n-1}-11a_{n-2}+6a_{n-3}$ with $a_0=2,a_1=5$ and $a_2=15$ **[November 2014]** **Solution:** The unique Solution to this recurrence relation and the given initial condition is the sequence $\{a_n\}$ with $a_n=1-2^n+2.3^n$
9(b)	**A factory makes custom sports cars at an interesting rate. In the first month only one car is made, in the second month two cars are made and so on, with n cars made in the nth month.** **(1) Set up recurrence relation for the number of cars produced in the first *n* months by this factory.** **(2) How many cars are produced in the first year?** **Solution:** (i) $a_n=n+a_{n-1}\quad,a_o=0(\because\ a_1=1,a_2=2+a_1,etc)$ (ii) By recursively $a_{12}=78$
10(a)	**Solve $S(n+1)-2S(n)=4^n$, with $S(0)=1$ and $n\geq1$** **Solution**: Given $a_{n+1}-2a_n-4^n=0$ Multiply by x^n, and sum over all $n=0$ to ∞.

	$\sum_{n=0}^{\infty} a_{n+1}x^n - 2\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} 4^n x^n = 0$ $G(x) = \frac{1-3x}{(1-2x)(1-4x)}$ By Applying Partial fractions we get $A = \frac{1}{2}, B = \frac{1}{2}$ $G(x) = \frac{1}{2}\sum_{n=0}^{\infty} 2^n x^n + \frac{1}{2}\sum_{n=0}^{\infty} 4^n x^n$ hence we get $a_n = 2^{n-1} + 2(4)^{n-1}$
10(b)	**Find the generating function of Fibonacci sequence.** **Solution** Fibonacci sequence : $f_n = f_{n-1} + f_{n-2},\ n \geq 2$ with $f_o = 0, f_1 = 1$ Multiply by z^n, and sum over all $n \geq 2$. $\sum_{n=2}^{\infty} f_n z^n = \sum_{n=2}^{\infty} f_{n-1} z^n + \sum_{n=2}^{\infty} f_{n-2} z^n$ $G(z) - f_0 - f_1 z = z(G(z) - f_0) + z^2(G(z))$ $G(z) = \sum_{n=0}^{\infty} f_n z^n$ Where $(i.e)\ G(z) - zG(z) - z^2G(z) = f_0 + f_1 z - zf_0$ $G(z) = \frac{z}{1-z-z^2}$
	UNIT III GRAPH THEORY
	PART – A
01.	**Define Graph.** **Ans:** A graph G = (V,E) consists of a finite non empty set V, the element of which are the vertices of G, and a finite set E of unordered pairs of distinct elements of V called the edges of G.
02.	**Define complete graph.** **Ans:** A graph of n vertices having each pair of distinct vertices joined by an edge is called a Complete graph and is denoted by K_n.
03.	**Define regular graph.** **Ans:** A graph in which each vertex has the same degree is called a regular graph. A regular graph has k – regular if each vertex has degree k.
04.	**Define Bipartite Graph with example.** **Ans:** Let G = (V,E) be a graph. G is bipartite graph if its vertex set V can be partitioned into two nonempty disjoint subsets V_1 and V_2 , called a bipartition, such that each edge has one end in V_1 and in V_2 . For eg C_6 

05.	**Define complete bipartite graph with example** **Ans:** A complete bipartite graph is a bipartite graph with bipartition V_1 and V_2 in which each vertex of V_1 is joined by an edge to each vertex of V_2. For eg. $K_{2,3}$ 
06.	**Define Subgraph.** **Ans:** A graph H = (V_1,E_1) is a subgraph of G = (V,E) provided that V_1 ,E_1 and for each e $\in$ E_1 , both ends of e are in V_1.
07.	**Define Isomorphism of two graphs.** **Ans:** Two graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ are the same or isomorphic, if there is a bijection $F : V_1 \to V_2$ such that $(u,v) \in E_1$ if and only if $(F(u), F(v)) \in E_2$.
08.	**Define strongly connected graph.** **Ans:** A digraph G is said to be strongly connected if for every pair of vertices, both vertices of the pair are reachable from one another.
09.	**State the necessary and sufficient conditions for the existence of an Eulerian path in a connected graph.** **Ans:** A connected graph contains an Euler path if and only if it has exactly two vertices of odd degree.
10.	**State Handshaking theorem.** **Ans:** If G = (V, E) is an undirected graph with e edges, then $\sum_i \deg(v_i) = 2e$
11.	**Define adjacency matrix.** **Ans:** Let G = (V,E) be a graph with n vertices . An "n x n" matrix A is an adjacency matrix for G if and only if for $i \leq I$, $j \leq n$, $A(i,j) = \begin{cases} 1 & \text{for}(i,j) \text{ in } E \\ 0 & \text{for}(i,j) \text{ is not in } E \end{cases}$
12.	**Define Connected graph.** **Ans:** A graph for which each pair of vertices is joined by a trail is connected.
13.	**Define Pseudo-graph.** **Ans:** A graph is called a pseudo-graph if it has both parallel edges and self loops.
14.	**Does there exist a simple graph with five vertices of the 0, 1, 2, 2, 3 degrees? If so, draw such a graph.** **Ans:**

	 Yes.
15.	**Draw a complete bipartite graph of $K_{2,3}$ and $K_{3,3}$** **Ans:** 
16.	**Define spanning subgraph.** **Ans:** Let a graph $H = (V_1,E_1)$ is a subgraph of $G = (V,E)$. H is a spanning subgraph of G if H is a subgraph of G with $V_1= V$ and $E_1 \subset E$.
17.	**Define Induced subgraph.** **Ans:** A graph $H = (V_1,E_1)$ is a subgraph of $G = (V,E)$. H is an induced subgraph of G such that E_1 consists of all the edges of G with both ends in V_1
18.	**Define Eulerian Circuit.** **Ans:** A circuit in a graph that includes each edge exactly once, the circuit is called an Eulerian circuit.
19.	**State the condition for Eulerian cycle.** **Ans:** (i) Starting and ending pts are same. (ii) Cycle should contain all edges of graph but exactly once
20	**Show that C_6 is a bipartite graph?** **Ans:** C_6 vertex set is partitioned into two set $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{ v_2, v_4, v_6\}$, where every edge of C_6 joins a vertex in V_1 to a vertex in V 
	PART - B
1(a)	**State and prove Handshaking Theorem.** If G = (V, E) is an undirected graph with e edges, then $\sum_i \deg(v_i) = 2e$ Proof: Since every edge is incident with exactly two vertices, every edge contributes 2 to the sum of the degree of the vertices. Therefore, all the e edges contribute ($2e$) to the sum of the degrees of the vertices. Hence $\sum_i \deg(v_i) = 2e$.

1(b)	**In any graph show that the number of odd vertices is even.** Let G = (V, E) be the undirected graph. Let v_1 and v_2 be the set of vertices of G of even and odd degrees respectively. Then by hand shaking theorem, $2e = \sum_{v_i \in v_1} \deg(v_i) + \sum_{v_j \in v_2} \deg(v_j)$. Since each deg($v_i$) is even, $\sum_{v_i \in v_1} \deg(v_i)$ is even. Since LHS is even, we get $\sum_{v_j \in v_2} \deg(v_j)$ is even. Since each deg(v_j) is odd, the number of terms contain in $\sum_{v_j \in v_2} \deg(v_j)$ or v_2 is even, that is, the number of vertices of odd degree is even.
2(a)	**Prove that a simple graph with at least two vertices has at least two vertices of same degree.** **Proof:** Let G be a simple graph with n $\geq$ 2 vertices. The graph G has no loop and parallel edges. Hence the degree of each vertex is $\leq$ n-1. Suppose that all the vertices of G are of different degrees. Following degrees 0, 1, 2, …, n-1 are possible for n vertices of G. Let u be the vertex with degree 0. Then u is an isolated vertex. Let v be the vertex with degree n-1 then v has n-1 adjacent vertices. Because v is not an adjacent vertex of itself, therefore every vertex of G other than u is an adjacent vertex of G. Hence u cannot be an isolated vertex, this contradiction proves that simple graph contains two vertices of same degree.
2(b)	**Prove that the maximum number of edges in a simple graph with n vertices is** $n_{c_2} = \frac{n(n-1)}{2}$ **Proof:** We prove this theorem, by the method of mathematical induction. For $n = 1$, a graph with 1 vertex has no edges. Therefore the result is true for n = 1. For n = 2, a graph with two vertices may have atmost one edge. Therefore 2 (2 –1) / 2 = 1. Hence for n = 2, the result is true. Assume that the result is true for n = k, i.e, a graph with k vertices has atmost $\frac{k(k-1)}{2}$ edges. Then for n = k + 1, let G be a graph having n vertices and G′ be the graph obtained from G, by deleting one vertex say, ‘v’ $\in$ V(G). Since G′ has k vertices then by the hypothesis, G′ has atmost $\frac{k(k-1)}{2}$ edges. Now add the vertex v to G′. ‘v’ may be adjacent to all the k vertices of G′. Therefore the total number of edges in G are $\frac{k(k-1)}{2} + k = \frac{k(k+1)}{2}$. Therefore the result is true for n = k+1. Hence, the maximum number of edges in a simple graph with ‘n’ vertices is $\frac{n(n-1)}{2}$.
3(a)	**Show that a simple graph G with *n* vertices is connected if it has more than** $\frac{(n-1)(n-2)}{2}$ ***edges*** **Proof:** Suppose G is not connected. Then it has a component of k vertices for some k, The most edges G could have is

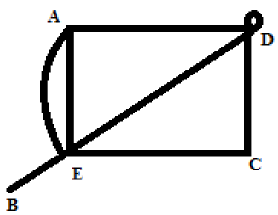
	$C(k,2)+C(n-k,2)=\dfrac{k(k-1)+(n-k)(n-k-1)}{2}$ $=k^2-nk+\dfrac{n^2-n}{2}$ This quadratic function of f is minimized at k = n/2 and maximized at k = 1 or k = n – 1 Hence, if G is not connected, then the number of edges does not exceed the value of this function at 1 and at n-1, namely $\dfrac{(n-1)(n-2)}{2}$.
3(b)	**If a graph G has exactly two vertices of odd degree, then prove that there is a path joining these two vertices.** **Proof:** Case (i): Let G be connected. Let v_1 and v_2 be the only vertices of G with are of odd degree. But we know that number of odd vertices is even. So clearly there is a path connecting v_1 and v_2, because G is connected. Case (ii): Let G be disconnected Then the components of G are connected. Hence v_1 and v_2 should belong to the same component of G. Hence, there is a path between v_1 and v_2.
4(a)	**Prove that a simple graph with n vertices and k components can have at most $\dfrac{(n-k)(n-k+1)}{2}$ edges.** Let the number of vertices of the ith component of G be $n_i, n_i \geq 1$.. $\sum_{i=1}^{k} n_i = n \Rightarrow \sum_{i=1}^{k}(n_i-1)=(n-k)$ Then $\Rightarrow \left(\sum_{i=1}^{k}(n_i-1)\right)^2 = n^2-2nk+k^2$ that is $\sum_{i=1}^{k}(n_i-1)^2 \leq n^2-2nk+k^2 \Rightarrow \sum_{i=1}^{k} n_i^2 \leq n^2-2nk+k^2+2n-k$ Now the maximum number of edges in the ith component of G = $\dfrac{n_i(n_i-1)}{2}=\dfrac{1}{2}\sum_{i=1}^{k} n_i^2-\dfrac{n}{2}$ $\leq \dfrac{(n^2-2nk+k^2+2n-k)}{2}-\dfrac{n}{2} \leq \dfrac{(n-k)(n-k+1)}{2}$
4(b)	**If all the vertices of an undirected graph are each of degree k, show that the number of edges of the graph is a multiple of k.** **Solution:** Let *2n* be the number of vertices of the given graph….(1) Let n_e be the number of edges of the given graph. By Handshaking theorem, we have $\sum_{i=1}^{2n} \deg v_i = 2n_e$ $2nk = 2n_e \quad (1)$ $n_e = nk$ Number of edges =multiple of k. Hence the number of edges of the graph is a multiple of k

<table>
<tr><td>5(a)</td><td>

Draw the graph with 3 vertices A,B,C, D & E such that the deg(A)=3,B is an odd vertex, deg(C)=2 and D and E are adjacent.

Solution:

d(E)=5 ,d(C)=2,d(D)=5 ,d(A)=3 d(B)=1



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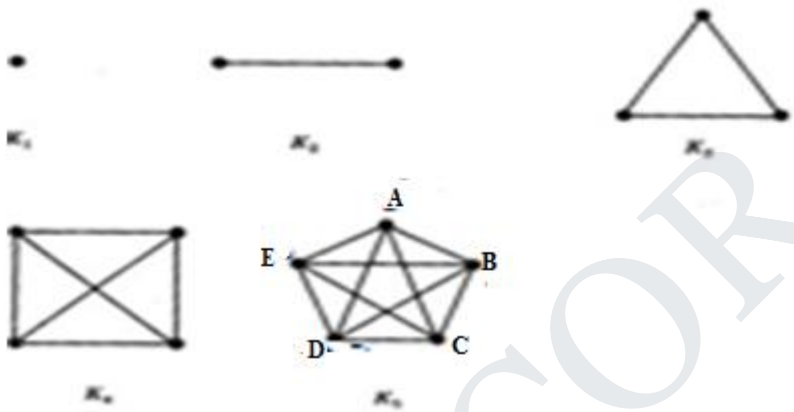
<tr><td>5(b)</td><td>

Draw the complete graph K_5 with vertices A,B,C,D,E. Draw all complete sub graph of K_5 with 4 vertices.

Solution:



complete sub graph with 4 vertices

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<tr><td>6(a)</td><td>

Prove that a given connected graph G is Euler graph if and only if all vertices of G are of even degree.

Solution:

Case (i) Prove If G is Euler graph→ Every vertex of G has even degree.

Case (ii) Prove If Every vertex of G has even degree.→ G is Euler graph (by Contradiction Method).

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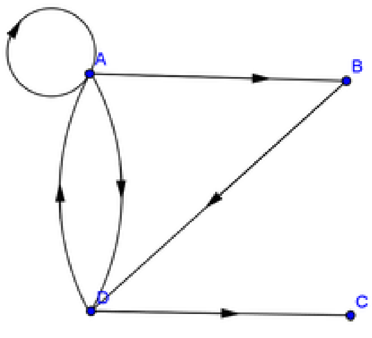
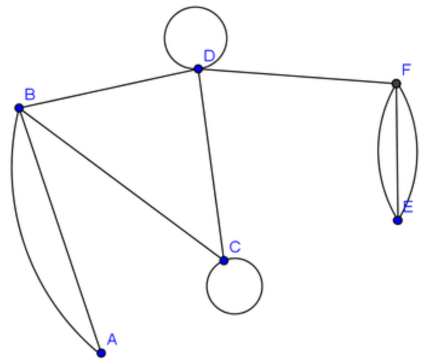
6(b) **Find the adjacency matrix of the given directed graph.**

(i)



(ii)



Answer:

$$(i)\quad \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \qquad (ii)\quad \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{pmatrix}$$

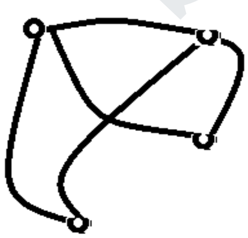
7(a) **Show that isomorphism of simple graphs is an equivalence relation.** **[November 2014]**

Solution:

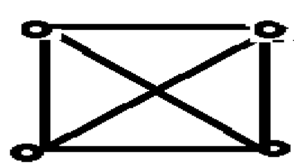
G is isomorphism to itself by the identity function, So isomorphism is reflexive. Suppose that G is isomorphic to H.Then there exists a one –to-one correspondence f from G to H that preserves adjacency and nonadjacency.It follows that f^{-1} is a one-to-one correspondence from H to G that preserves adjacency and non-adjacency.Hence isomorphism is symmetric.If G is isomorphic to H and H is isomorphic to K then there are ono-to-one correspondences f and g from G to H and from H to K that preserves adjacency and nonadjacency.It follows that $g\,o\,f$ is a one-to-one correspondencies from G to K that preserves adjacency and non-adjacency.Hence isomorphism is transitive.

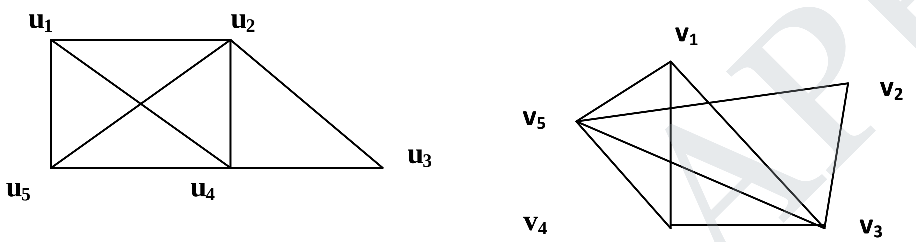
7(b) **Find the incidence matrix for the following graph.**

(i)



(ii)



	Answer: $(i)\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \quad (ii)\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$
8(a)	**Examine whether the following pair of graphs are isomorphic. If not isomorphic, give the reasons**  **Solution:** Same number of vertices and edges. Also an equal number of vertices with a given degree. The adjacency matrices of the two graphs are $\begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$ since the two adjacency matrices are the same, the two graphs are isomorphic.
8(b)	**Prove that if a graph G has not more than two vertices of odd degree, then there can be Euler path in G.** **Statement**: Let the odd degree vertices be labeled as V and W in any arbitrary order. Add an edge to G between the vertex pair (V,W) to form a new graph G'. Now every vertex of G' is of even degree and hence G' has an Eulerian Trail T. If the edge that we added to G is now removed from T, It will split into an open trail containing all edges of G which is nothing but an Euler path in G
9(a)	**Show that K_7 has Hamiltonian graph. How many edge disjoint Hamiltonian cycles are there in K_7 ? List all the edge-disjoint Hamiltonian cycles. Is it Eulerian graph ?**

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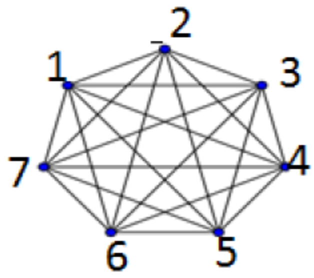
Solution: The Graph of K_7



K_7 has two edges disjoint Hamiltonian cycles.
The edge disjoint Hamiltonian cycles are
1_2_3_4_5_6_7_1 and 1_3_6_2_4_7_5_1
K_7 is an Eulerian graph

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<tr><td>9(b)</td><td>

Let G be a simple indirected graph with n vertices. Let u and v be two non adjacent vertices in G such that deg(u) + deg(v) ≥ n in G. Show that G is Hamiltonian if and only if G + uv is Hamiltonian.
Solution:
If G is Hamiltonian, then obviously G + uv is also Hamiltonian.
Conversely, suppose that G + uv is Hamiltonian, but G is not. Then by Dirac theorem, we have
deg(u) + deg(v) < n
which is a contradiction to our assumption.
Thus G + uv is Hamiltonian implies G is Hamiltonian.

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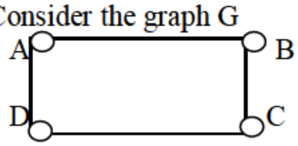
<tr><td>10(a)</td><td>

Draw a graph that is both Eulerian and Hamiltonian.
Solution:
Example of Eulerian and Hamiltonian.

Consider the graph G



n G, consider the cycle A-B-C-D-A. Since the cycle contains all the edges, G is Eulerian. Moreover, since the cycle contains all the vertices exactly once, G is Hamiltonian.

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<tr><td>10(b)</td><td>

Prove that any 2 simple connected graphs with *n* vertices all of degree 2 are isomorphic.
Proof:
We know that total degree of a graph is given by

$$\sum_{i=1}^{n} d(V_i) = 2|E|$$

Then |V| = number of vertices n
|E| = number of edges
Further the degree of every vertex is 2. Therefore we have,

$$\sum_{i=1}^{n} 2 = 2|E|$$

$$2((n) - 1 + 1) = 2|\text{E}|$$

$$\Rightarrow \text{n} = |\text{E}|$$

Hence number of edges = number of vertices. Hence they are isomorphic.

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</table>

	UNIT – IV GROUP THEORY
	PART – A
01.	**Define Algebraic system.** **Ans:** A set together with one or more n-ary operations on it is called an algebraic system. Example (Z,+) is an algebraic system.
02.	**Define Semi Group.** **Ans:** Let S be non empty set, * be a binary operation on S. The algebraic system (S, *) is called a semi group, if the operation is associative. In other words (S,*) is a semi group if for any x, y, z $\in$ S, x* (y * z) = (x* y)* z.
03.	**Define Monoid.** **Ans:** A semi group (M, *) with identity element with respect to the operation * is called a Monoid. In other words (M,*) is a semi group if for any x, y, z $\in$ M, x* (y * z) = (x* y)* z and there exists an element e $\in$ M such that for any x $\in$ M then e* x = x* e = x.
04.	**Define Group.** **Ans:** An algebraic system (G,*) is called a group if it satisfies the following properties: (i) G is closed with respect to * (ii) * is associative. (iii) Identity element exists. (iv) Inverse element exists.
05.	**State any two properties of a group.** **Ans:** (i)The identity element of a group is unique. (ii) The inverse of each element is unique.
06.	**Define a Commutative ring.** **Ans:** If the Ring (R, *) is commutative, then the ring (R, +, *) is called a commutative ring.
07.	**Show that the inverse of an element in a group (G, *) is unique.** **Ans:** Let (G,*) be a group with identity element e. Let ‘b’ and ‘c’ be inverses of an element ‘a’ a * b = b * a = e, a * c = c * a = e. b = b * e = b * (a * c) = (b * a) * c = e * c = c b = c. Hence inverse element is unique.
08.	**Give an example of semi group but not a Monoid.** **Ans:** The set of all positive integers over addition form a semi-group but it is not a Monoid.
09.	**Prove that the semigroup homomorphism preserves idempotency**. **Ans:** Let a $\in$ S be an idempotent element. $\therefore \quad a * a = a$ $g(a * a) = g(a)$ $g(a) \circ g(a) = g(a)$ This shows that $g(a)$ is an idempotent element in T. Therefore the property of idempotency is preserved under semigroup homomorphism.
10.	**Define cyclic group.** **Ans:** A group (G,*) is said to be cyclic if there exists an element a $\in$ G such that every element of G can be written as some power of ‘a’.

11.	**Define group homomorphism.** **Ans:** Let (G,*) and $(S,\circ)$ be two groups. A mapping f : G → S is said to be a group homomorphism if for any a, b ∈ G f(a*b) = f(a) ∘ f(b).
12.	**Define Left Coset.** **Ans:** Let (H,*) be a subgroup of (G,*). For any a ∈ G the set H is defined by aH = {a*h: h ∈ H} is called the right coset of H determined by a ∈ G.
13.	**State Lagrange's theorem.** **Ans:** The order of the subgroup of a finite group G divides the order of the group.
14.	**Define Ring.** **Ans:** An algebraic system (R, +, *) is called a ring if the binary operations + and R satisfies the following. (i) (R,+) is an abelian group (ii) (R,*) is a semi group (iii) The operation is distributive over +.
15.	**Define field.** **Ans:** A commutative ring (F, +, *) which has more than one element such that every nonzero element of F has a multiplicative inverse in F is called a field.
16.	**Define Integral Domain.** **Ans:** A commutative ring R with a unit element is called an integral domain if R has no zero divisors.
17.	**Let T be the set of all even integers. Show that the semi groups (Z,+) and (T,+) are isomorphic.** **Ans:** Define a function f: Z → T by f(n) = 2n where n_1, n_2 ∈ N. f is a homomorphism since f(n_1+ n_2) =f n_1) +f(n_2). f is one-one since f(n_1) = f(n_2). f is onto since f(a) = 2a. therefore f is an isomorphism.
18.	**Show that the semi group homomorphism preserves the property of idempotency.** **Ans:** Let f : (M,*) → (H,Δ) be a semi group homomorphism. x is idempotent element in M. x*x = x. f(x*x) = f(x) Δ f(x).
19.	**Let <M, *, e_M> be a Monoid and a∈M. If a is invertible, then show that its inverse is unique.** **Ans:** Let 'b' and 'c' be inverses of 'a'. Then a * b = b * a = e and a * c = c * a = e. Now b = b * e = b * (a * c) = (b * a) * c = e * c = c.
20.	**If H is a subgroup of the group G, among the right cosets of H in G , prove that there is only one subgroup H.** **Ans:** Let Ha be a right coset of H in G where a ∈ G. If Ha is a subgroup of G, then e ∈ Ha where e is the identity element in G.Ha is an equivalence class containing a with respect to equivalence relation. So that e ∈ Ha => He = Ha. So Ha =H.
	PART – B
1(a)	**Show that group homomorphism preserves identity, inverse and subgroup.** **Proof:** **Identity** Let $g:(G,*)\rightarrow(H,\Delta)$ be a group homomorphism. Now $g(e_G)=g(e_G*e_G)=g(e_G)\,\Delta\, g(e_G)$ Hence $g(e_G)$ is an idempotent element and $g(e_G)=e_H$ is the identity element.

	Inverse $g(a*a^{-1}) = g(e_G) = g(a^{-1}*a)$ $g(a)\Delta g(a^{-1}) = e_H = g(a^{-1})\Delta g(a)$ Hence $g(a^{-1})$ is the inverse of $g(a)$ **subgroup** Let S be the subgroup of $(G,*)$ (i) As $e_G \in S$ then $e_H \in g(S)$ (ii) If x = g(a) $\in S$ then $x^{-1} = [g(a)]^{-1} \in g(S)$ (iii) If a, b $\in S$ then g(a*b) $g(a*b) = g(a)\Delta g(b) = x\Delta y \in g(S)$ Hence $g(S)$ is a subgroup of H.
1(b)	**Let (S, *) be a semi-group. Prove that there exists a homomorphism g: S → S^S. where < S^S, ∘> is a semi-group of functions from S to S under the operation f (left) composition.** **Solution:** For any element a ∈ S, let g(a) = f_a, where $f_a \in S^S$ and f_a is defined by $f_a(b)$ = a * b for any b ∈S. Now g(a * b) = f_{a*b}, where f_{a*b} (c) = (a * b) * c = a * (b* c) = $f_a(f_b(c))$ = $(f_a \circ f_b)$ (c) Therefore, g(a* b) = $f_{a*b} = f_a \circ f_b$ = g(a) ∘ g(b). Hence g is a homomorphism. For an element a ∈ S, the function f_a is completely determined from the entries in the row corresponding to a in the composition table of (S, *). Since f_a = g(a), every row of the table determines the image under the homomorphism of g.
2(a)	**Show that the set N of natural numbers is a semigroup under the operation x * y = max {x, y}. Is it a Monoid?** **Proof:** Clearly if x, y ∈ N then max{x,y} = x or y ∈ N. Hence closure is true. Now (x*y)*z = max {x*y, z} = max {x,y*z} = x*(y*z). Hence N is associative. e = ∞ is the element in N such that x*e=e*x=e. Hence (N, *, ∞) is Monoid.
2(b)	**Prove that if (G, *) is an Abelian group, if and only if $(a * b)^2 = a^2 * b^2$** **Proof:** Let G be an abelian group. Now $(a * b)^2$ = (a * b) * (a * b) = a * (b * a) * b = a * (a * b) * b = $a^2 * b^2$. Conversely, let $(a * b)^2 = a^2 * b^2$ (a* b) * (a*b) = (a * a) * (b * b) ⇒ $(a^{-1} * a) * (b * a) * (b * b^{-1}) = (a^{-1} * a) * a * b * (b * b^{-1})$ ⇒ b * a = a * b. Hence G is abelian.
3(a)	**Prove that the necessary and sufficient condition for a non empty subset H of a group (G, *) to be a subgroup of G if $a, b \in H \Rightarrow a * b^{-1} \in H$** **Proof:** **Necessary Condition:** Let us assume that H is a subgroup of G. Since H itself a group, we have if a, b ∈ H implies a*b ∈ H

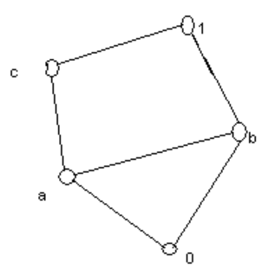
	Since b $\in$ H then $b^{-1} \in$ H which implies $a * b^{-1} \in$ H. **Sufficient Condition:** Let $a * b^{-1} \in$ H, for a, b $\in$ H If a $\in$ H, which implies $a * a^{-1} = e \in H$ Hence the identity element 'e' $\in$ H. Let a, e $\in$ H, then $e * a^{-1} = a^{-1} \in H$ Hence a^{-1} is the inverse of 'a'. Let a, $b^{-1} \in$ H, then $a * (b^{-1})^{-1} = a * b \in H$. Therefore H is closed and clearly * is associative. Hence H is a subgroup of G.
3(b)	**Prove that intersection of two subgroups is a subgroup, but their union need not be a subgroup of G.** **Proof:** Let A and B be two subgroups of a group G. we need to prove that $A \cap B$ is a subgroup. i.e. it is enough to prove that $A \cap B \neq \phi$ and $a, b \in A \cap B \Rightarrow a * b^{-1} \in A \cap B$. Since A and B are subgroups of G, the identity element e $\in$ A and B. $\therefore A \cap B \neq \phi$ $Let \; a, b \in A \cap B \Rightarrow a, b \in A \; and \; a, b \in B$ $\Rightarrow a * b^{-1} \in A \; and \; a * b^{-1} \in B \Rightarrow a * b^{-1} \in A \cap B$ Hence $A \cap B$ is a subgroup of G. Consider the following example, Consider the group, (Z, +), where Z is the set of all integers and the operation + represents usual addition. Let $A = 2Z = \{0, \pm 2, \pm 4, \pm 6, ...\}$ and $B = 3Z = \{0, \pm 3, \pm 6, \pm 9, ...\}$. (2Z, +) and (3Z, +) are both subgroups of (Z, +). Let $H = 2Z \cup 3Z = \{0, \pm 2, \pm 3, \pm 4, \pm 6 ....\}$ Note that $2, 3 \in H$, $but \; 2 + 3 = 5 \notin H \Rightarrow 5 \notin 2Z \cup 3Z$ i.e $2Z \cup 3Z$ is not closed under addition. Therefore $2Z \cup 3Z$ is not a group i.e. $2Z \cup 3Z$ is not a subgroup of (Z, +). Therefore (H, +) is not a subgroup of (Z, +).
4(a)	**Show that the Kernel of a homomorphism of a group (G, *) into another group (H, Δ) is a subgroup of G.** **Proof:** Let K be the Kernel of the homomorphism g. That is $K = \{x \in G \mid g(x) = e'\}$ where e' the identity element of H. is Let x, y $\in$ K. Now $g(x * y^{-1}) = g(x) \Delta g(y^{-1}) = g(x) \Delta [g(y)]^{-1} = e' \Delta (e')^{-1} = e' \Delta e' = e'$ $x * y^{-1} \in K$ Therefore K is a subgroup of G.
4(b)	**State and prove Cayley's theorem on permutation groups.** **Statement:** Every finite group of order "n" is isomorphic to a group of degree n.

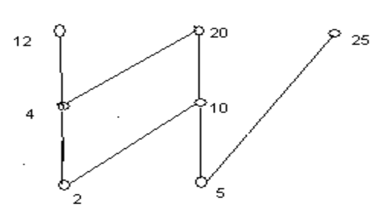
	Proof: Let G be the given group and A(G) be the group of all permutations of the set G. For any $a \in G$, define a map $f : G \to G$ such that f(x) = ax and we have to prove the following things (i) f_a is well defined. (ii) f_a is one – one (iii) f_a is onto Now let K be the set of all permutations and define a map $\chi : G \to K$ such that $\chi(a) = f_a$ Clearly χ is one-one, onto and homomorphism and hence χ is isomorphism which proves the theorem.
5(a)	**Prove that every subgroup of a cyclic group is cyclic.** **Proof:** Let (G,*) be the cyclic group generated by an element $a \in$ G and let H be the subgroup of G. If H contains identity element alone, then trivially H is cyclic. Suppose if H contains the element other than the identity element. Since $H \subseteq G$, any element of H is of the form a^k for some integer k. Let "m" be the smallest positive integer such that $a^m \in$ H. Now by division algorithm theorem we have $k = qm + r$ where $0 \leq r < m$. Now $a^k = a^{qm+r} = (a^m)^q \cdot a^r$ and from this we have $a^r = (a^m)^{-q} \cdot a^r$. Since $a^m, a^k \in$ H, we have $a^r \in$ H. Which is a contradiction that $a^m \in$ H such that "m: is small. Therefore r = 0 and $a^k = (a^m)^q$. Thus every element of H is a power of a^m and hence H is cyclic.
5(b)	**Prove that every cyclic group is an Abelian group.** **Proof:** Let (G,*) be the cyclic group generated by an element $a \in$ G. Then for any two element $x, y \in$ G, we have $x = a^n$, $y = a^m$, where m, n are integer. Now $x*y = a^n * a^m = a^{n+m} = a^{m+n} = a^m * a^n = y * x$
Hence	(G,*) is abelian.
6(a)	**State and Prove Lagrange's theorem** **Statement:** The order of each subgroup of a finite group is divides the order of the group. **Proof:** Let G be a finite group and o(G) = n and let H be a subgroup of G and o(H) = m. For $x \in G$, the right coset of H_x is defined by $H_x = \{h_1 x, h_2 x, h_3 x, \ldots\ldots h_m x\}$. Since there is a one to one correspondence between H and H_x, the members of H_x are distinct. Hence, each right coset of H in G has 'm' distinct members. We know that any two right cosets of H in G are either identical or disjoint. i.e. let H be a subgroup of a group G. *Let* $x, y \in G$. Let H_x and H_y be two right cosets of H in G. we need to prove that either $H_x = H_y$ or $H_x \cap H_y = \phi$. Suppose $H_x \cap H_y \neq \phi$. Then there exists an element $H_x \cap H_y$ Thus by proving O(G)/O(m)=k O(H) is a divisor of O(G) → O(H) divides O(G).
6(b)	**Let (G, *) and (H, Δ) be groups and $g : G \to H$ be a homomorphism. Then the Kernal of g is a normal subgroup.** **Proof:** Let K be the Kernel of the homomorphism g. That is $K = \{x \in G \mid g(x) = e'\}$ where e' the identity element of H. is Let $x, y \in$ K. Now

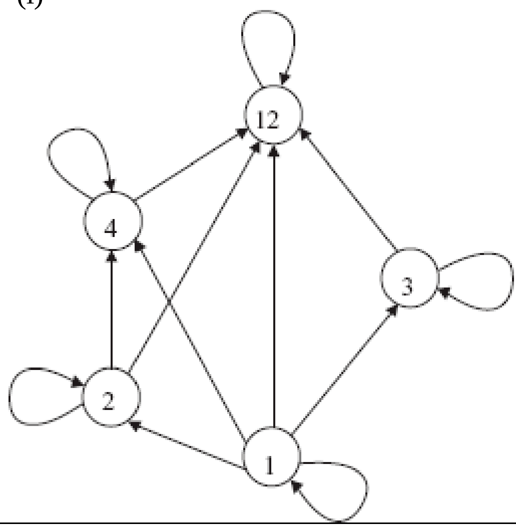
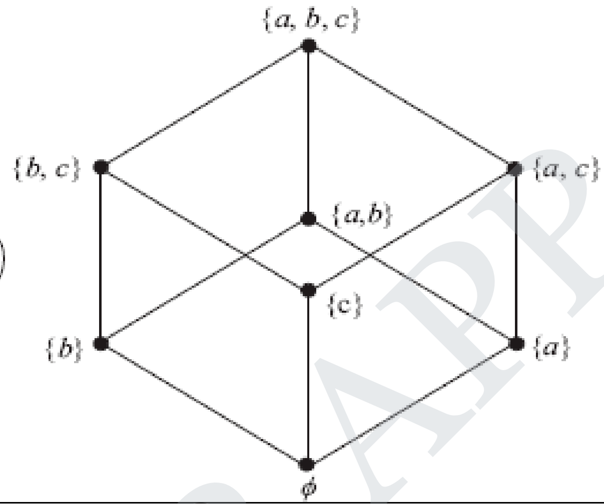
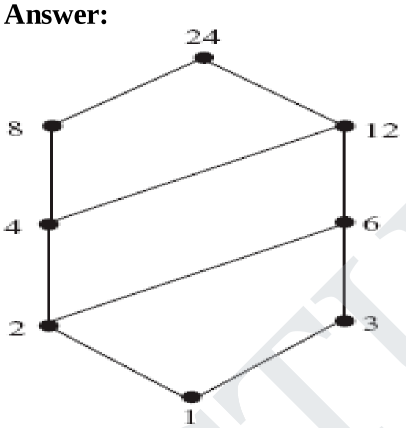
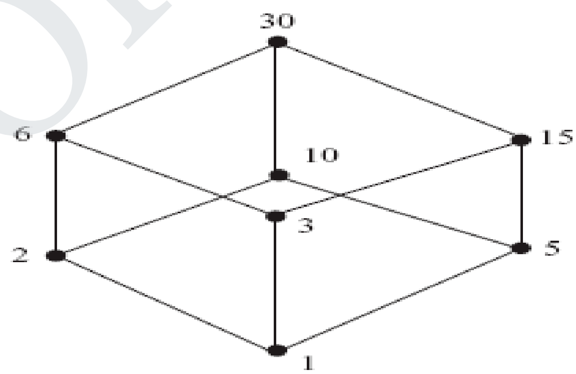
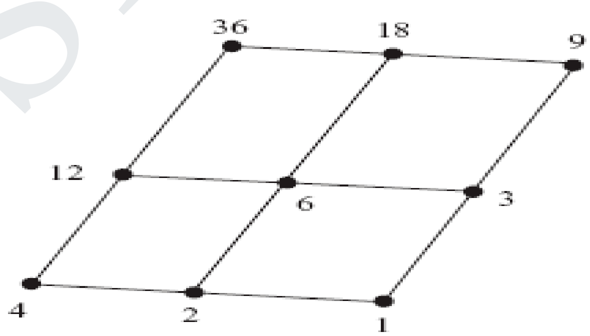
	$g(x*y^{-1}) = g(x)\,\Delta\, g(y^{-1}) = g(x)\,\Delta\,[g(y)]^{-1} = e'\,\Delta\,(e')^{-1} = e'\,\Delta\, e' = e'$ $x*y^{-1} \in K$ Therefore K is a subgroup of G. Let $x \in K,\ f \in G$ $g(f*x*f^{-1}) = g(f)*g(x)*g(f^{-1}) = g(f)\ e'\,[g(f)]^{-1} = g(f)\,[g(f)]^{-1} = e'$ $\therefore\quad f*x*f^{-1} \in K$ Thus K is a normal subgroup of G.
7(a)	**State and prove the fundamental theorem of group homomorphism** **Statement:** If f is a homomorphism of G onto G' with kernel K, then $G/K \cong G'$. **Proof:** Let $f: G \to G'$ be a homomorphism. Then K = Ker (f) = $\{x \in G \mid f(x) = e'\}$ is a normal subgroup and also the quotient set $(G/K, \otimes)$ is a group. Define $\phi: G/K \to G'$ given by $\phi(Ka) = f(a)$. Now we have to prove (i) ϕ is well defined. (ii) ϕ is a homomorphism. (iii) ϕ is one – one. (iv) ϕ is onto. From this proof's we have $G/K \cong G'$
7(b)	**Prove that intersection of any two normal subgroups of a group (G, *) is a normal subgroup of a group (G, *)** **Proof:** Let G be the group and H and K are the subgroups of G. Since H and K are subgroups of G, $e \in H$ and $e \in K \Rightarrow e \in H \cap K$. Thus H$\cap$ K is nonempty. Since $ab^{-1} \in H$ and $ab^{-1} \in K \Rightarrow ab^{-1} \in H \cap K$ Since $gxg^{-1} \in H$ and $gxg^{-1} \in K \Rightarrow gxg^{-1} \in H \cap K$ Thus $H \cap K$ is a Normal subgroup of G.
8(a)	**Prove that every subgroup of an Abelian group is a normal subgroup.** **Proof:** Let (G,*) be an abelian group and (N,*) be a subgroup of G. Let g be an element of G and n be an element of N. Now $g*n*g^{-1} = (n*g)*g^{-1} = n*(g*g^{-1}) = n*e = n \in N$ Hence for all $g \in G$ and $n \in N,\ g*n*g^{-1} \in N$ Therefore (N,*) is a normal subgroup
8(b)	**Prove that a sub group H of a group is normal if $x*H*x^{-1} = H,\ \forall\ x \in G$** **Proof:** $Let\ x*h*x^{-1} = H$

	$\Rightarrow \quad x * H * x^{-1} \subseteq H, \quad \forall\ x \in G$ $\Rightarrow$ H is a normal subgroup of G. Conversely, let us assume that H is normal subgroup of G. $x * H * x^{-1} \subseteq H, \quad \forall\ x \in G$ **Now** $x \in G \Rightarrow x^{-1} \in G$ $i.e. \quad x^{-1} * H * \left(x^{-1}\right)^{-1} \subseteq H, \quad \forall\ x \in G$ $x^{-1} * H * x \subseteq H$ $x * (x^{-1} * H * x) * x^{-1} \subseteq x * H * x^{-1}$ $e * H * e \subseteq x * H * x^{-1}$ $H \subseteq x * H * x^{-1}$ $\therefore \quad x^{-1} * H * x = H$
9(a)	**Prove that every subgroup of a cyclic group is normal.** **Proof:** We know that every cyclic group is Abelian. That is x * y = y * x. Let G be the cyclic group and let H be a subgroup of G. Let $x \in$ G and $h \in H$ then $x * h * x^{-1} = x * \left(h * x^{-1}\right) = x * \left(x^{-1} * h\right) = (x * x^{-1}) * h = e * h = h \in H$ Thus for $x \in G$ and $h \in H$, $x * h * x^{-1} \in H$ Thus H is a normal subgroup of G. Therefore every subgroup of a cyclic group is normal
9(b)	**Prove that every field is an integral domain, but the converse need not be true.** **Proof:** Let $(F, +, \cdot)$ be a field. That is F is a commutative ring with unity. Now to prove F is an integral domain it is enough to prove it has non-zero divisor. Let $a, b \in F$ such that a . b = 0 and let $a \neq 0$ then $a^{-1} \in F$ Now $a^{-1} \cdot (a \cdot b) = (a^{-1} \cdot a) \cdot b$ $a^{-1} \cdot 0 = 1 \cdot b$ $0 = b.$ Therefore F has non-zero divisor
10(a)	**If R is a commutative ring with unity whose ideals are {0} and R, then prove that R is a field.** **Proof:** We have to show that for any $0 \neq a \in R$ there exists an element $0 \neq b \in R$ such that ab = 1. Let $0 \neq a \in R$ Define $Ra = \{ra \mid r \in R\}$ Proof of Ra is an ideal Since $e \in R \Rightarrow ea \Rightarrow Ra \Rightarrow a \in Ra$ $\therefore\ Ra \neq \{0\}$ (since $a \neq 0$)

	Therefore the hypothesis of the theorem Ra = R This means that every element of R is a multiple of 'a' by some element of R. $\forall x \in R,\ x = ra,\ for\ some\ \ r \in R$ $For\ \ I \in R$ $1 = ba,\ for\ some\ \ 0 \neq b \in R$ $\therefore\ \ ab = 1$
10(b)	**Prove that $\{Z_p, +_p, *_p\}$ is an integral domain if and only if p is prime.** **Solution:** Let us assume that Z_p be an integral domain and to prove that p is prime. Suppose p is not prime then p = mn, where 1< m < p, 1 < n < p. Hence mn = 0. Therefore 'm' and 'n' are zero divisors and hence Z_p is not an integral domain. Which is a contradiction. Hence p is a prime. Conversely, Suppose p is prime. Let $a, b \in Z_p$ and ab = 0 Then ab = pq where $q \in Z_p$ then p divides ab i.e p divides a (or) p divides b therefore a = 0 (or) b = 0 thus Z_p has no zero divisors. Also Z_p is a commutative ring with identity. Hence Z_p is an integral domain.
	UNIT-V LATTICES AND BOOLEAN ALGEBRA
	PART – A
01.	**Define lattice.** **Ans:** A partially ordered set (L,≤) in which every pair of elements has a least upper bound and greatest lower bound is called a lattice.
02.	**Define lattice homomorphism and isomorphism.** **Ans:** If $(L_1, \wedge, \vee)\ and\ (L_2, \oplus, *)$ are two lattices, a mapping $f: L_1 \to L_2$ is called a lattice homomorphism from L_1 to L_2 , if for any $a, b \in L_1$, $f(a \vee b) = f(a) \oplus f(b)\ and\ f(a \wedge b) = f(a) * f(b)$. If a homomorphism $f: L_1 \to L_2$ of two lattices $(L_1, \wedge, \vee)\ and\ (L_2, \oplus, *)$ is objective i.e. one -one, onto, then f is called an isomorphism.
03.	**Define sub lattice with example.** **Ans:** A non-empty subset M of a lattice $(L, \wedge, \vee)$ is called a sub lattice of L, if and only if M is closed under both the operations $\wedge\ and\ \vee$ that is if a, b∊ M, then $a \vee b\ and\ a \wedge b$ also in M. (S_n, D) is a sub lattice of (Z_+, D)
04.	**Define partial ordering on S.** **Ans:** A relation $\leq$ on a set S is called a partial ordering on S if it has the following three properties S is reflexive, anti-symmetric, transitive. A set S together with a partial ordering is called a partially ordered set or poset.

05.	**Define Hasse diagram.** **Ans:** Hasse diagram of a finite partially ordered set S is the directed graph whose vertices are the elements of S and there is a directed edge from a to b whenever a < b in S.
06.	**Simplify the Boolean expression** $a'.b'.c + a.b'.c + a.b'.c'$ **, using Boolean algebraic identities.** **Ans:** $a'.b'.c + a.b'.c + a.b'.c' = a'.b'.c + a.b'.(c + c') = a'.b'.c + a.b'.1 = b'.(a + a'.c) =$ $b'.(a + a')(a.c) = a.b' + b'.c$
07.	**Prove that** $D_{42} \equiv \{S_{42}, D\}$ **is a complemented lattice by finding the complements of all the elements.** **Ans:** $D_{42} = \{1, 2, 3, 4, 7, 14, 21, 42\}$ The complement of 1 is 42, the complement of 2 is 21, the complement of 3 is 14, the complement of 6 is 7, the complement of 14 is 3, the complement of 21 is 2, the complement of 42 is 1. the complement of 7 is 6. Every element has a complement. Hence $D_{42} \equiv \{S_{42}, D\}$ is a complemented lattice
08.	**In the poset** $(Z^+, /)$**, are the integers 3 and 9 comparable? Are 5 and 7 comparable?** **Ans:** Since 3/9, the integers 3 and 9 are comparable. For 5, 7 neither 5/7 nor 7/5. Therefore, the integers 5 and 7 are not comparable.
09.	**When a lattice is called complete?** **Ans:** A lattice <L, *, ⊕> is called complete if each of its non-empty subsets has a least upper bound and a greatest lower bound.
10.	**Define direct product of lattice.** **Ans:** Let $(L, *, \oplus)$ and $(S, \wedge, \vee)$ be two lattices. The algebraic system $(L \times S, \bullet, +)$ in which the binary operation + and $\bullet$ on L x S are such that for any (a_1, b_1) and (a_2, b_2) in L x S $(a_1, b_1).(a_2, b_2) = (a_1 * a_2, b_1 \wedge b_2)$ $(a_1, b_1) + (a_2, b_2) = (a_1 \oplus a_2, b_1 \vee b_2)$ is called the Direct product of the lattice $(L, *, \oplus)$ and $(S, \wedge, \vee)$.
11.	**Prove that** $a + \bar{a}b = a + b$ **Ans:** $a + \bar{a}b = a + ab + \bar{a}b$ (a = a + ab) $= a + b(a + \bar{a}) = a + b$ Since $b \wedge c = a$ and $b \vee c = 1$ $b \wedge a = a$ and $b \vee a = b$ Therefore b does not have any complement .the given lattice is not complemented lattice.
12.	**Check the given lattice is complemented lattice or not.** **Ans:** 

13.	**Reduce the expression** $a.\bar{a}b$ **.** **Ans:** $a.\bar{a}b$ =0.b = 0
14.	**Prove the involution law** $(a')' = a$ **.** **Ans:** It is enough to show that $a' + a = 1 \ and \ a.a' = 0$ By dominance laws of Boolean algebra, we get $a' + a = 1 \ and \ a.a' = 0$ By commutative laws, we get $a + a' = 1 \ and \ a'.a = 0$. Therefore Complement of a' is a $(a')' = a$
15.	**Determine whether the following posets are lattices.** **(i) ({1,2,3,4,5},/)** **(ii) ({1,2,4,8,16},/)** **Ans:** ({1,2,3,4,5},/) is not a lattice because there is no upper bound for the pairs (2,3) and (3,5). (ii) ({1,2,4,8,16},/) is a lattice. Since every pair has a LUB and a GLB .
16.	**Reduce the expression a(a+c).** **Ans:** a(a+c)= aa+ac = a+ac = a(1+c) = a.
17	**Show that the 'greater than or equal to 'relation (≥) is a partial ordering on the set of integers.** **Ans:** Since a≥ a for every integer a, ≥ is reflexive. If a≥ b and b≥a,then a=b.hence '≥' is antisymmetric. Since a≥b and b≥c imply that a≥c,' ≥' is transitive. Therefore '≥' is a partial order relation on the set of integers.
18.	**Prove that any lattice homomorphism is order preserving.** **Ans:** Let $f : L_1 \rightarrow L_2$ be a homomorphism. Let $a \leq b$ Then GLB$\{a,b\}$= $a \wedge b$ =a, LUB$\{a,b\}$= $a \vee b$ =b Now $f(a \wedge b) = f(a) \Rightarrow f(a) \wedge f(b) = f(a)$ i.e., GLB $\{f(a), f(b)\} = f(a)$. Therefore $f(a) \leq f(b)$ If $a \leq b$ implies $f(a) \leq f(b)$. Therefore f is order preserving.
19.	**Is the poset** $(Z^+, /)$ **a lattice.** **Ans:** Let a and b be any 2 positive integer. Then LUB{a,b} =LCM {a,b} and GLB{a,b}= GCD{a,b} should exists in Z^+. For example, let a=4, b=20 Then LUB{a,b} = lcm {4,20} = 1 and GLB{a,b}=gcd{4,20}=4 Hence, both GLB and LUB exist. Therefore The poset $(Z^+, /)$ is a lattice.
20.	**Which elements of the poset ({2,4,5,10,12,20,25},/) are maximal and which are minimal?** **Ans:** The relation R is R={(2,4) (2,10) (2,12) (2,20) (4,12) (4,20) (5,10) (5,20) (5,25) (10,20)} Its Hasse diagram is  The maximal elements are 12, 20, and 25 and The minimal elements are 2 and 5.

	PART – B
1(a)	**Draw the Hasse diagram for (i) $P_1 = \{1, 2, 3, 4, 12\}$ and $\leq$ is a relation such that $x \leq y$ if x divides y (ii) Let $S = \{a, b, c\}$ and $\tilde{A} = P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ Consider the partial order of set inclusion ($\subseteq$).** **Answer:** (i)  (ii) 
1(b)	**Draw the Hasse diagram for $D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$, $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$, $D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$ considering the partial order divisibility.** **Answer:**  D_{24}  D_{30}  D_{36}
2(a)	**Show that every chain is a distributive lattice.** **Solution:**

	Let $(L,\leq)$ be a chain and let a, b, c $\in L$, then $a \leq b \leq c$ and $a \geq b \geq c$. When $a \leq b \leq c$, we have $a \wedge (b \vee c) = a \wedge c = a$ also, $(a \wedge b) \vee (a \wedge c) = a \vee a = a$. Thus $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. Again $a \vee (b \wedge c) = a \vee b = b$. Also $(a \vee b) \wedge (a \vee c) = b \wedge c = b$ therefore $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$. $\Rightarrow (L,\vee,\wedge)$ is a distributive lattice. When $a \geq b \geq c$, we have $a \wedge b = b$ and $a \vee b = a$ Now, $a \wedge (b \vee c) = a \wedge b = b$ $(a \wedge b) \vee (a \wedge c). = b \vee c = b$ $\therefore\ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. $Also$, $a \vee (b \wedge c) = (a \vee c) = a$ $(a \vee b) \wedge (a \vee c) = (a \wedge a) = a$ $\therefore\ a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ Hence, $(L,\vee,\wedge)$ is a distributive lattice. This indicates that every chain is a distributive lattice.
2(b)	**State and prove Isotonicity property in lattice.** **Statement:** Let $(L,\wedge,\vee)$ be given Lattice. For any a, b, c $\in L$, we have, $b \leq c \ \Rightarrow$ 1) $a \wedge b \leq a \wedge c$ 2) $a \vee b \leq a \vee c$ **Proof:** Given $b \leq c$ Therefore $GLB\{b,c\} = b \wedge c = b$ and $LUB\{b,c\} = b \vee c = c$ Claim 1: $a \wedge b \leq a \wedge c$ To prove the above, it's enough to prove $GLB\{a \wedge b\ ,\ a \wedge c\} = a \wedge b$ Claim 2: $a \vee b \leq a \vee c$ To prove the above it's enough to prove $LUB\{a \vee b\ ,\ a \vee c\} = a \vee c$
3(a)	**Prove that the De Morgon's laws hold good for a complemented distributive lattice** $(L,\wedge,\vee)$ **.** **Solution:** The De Morgon's Laws are (1) $(a \vee b)' = a' \wedge b'$ (2) $(a \wedge b)' = a' \vee b'$, for all $a, b \in B$ Proof: Let (L, $\wedge$, $\vee$) be a complemented distributive lattice. Let a, b $\in$ L. Since L is a complemented lattice, the complements of 'a' and 'b' exist. Let the complement a be a' and the complement of b be b' Now

	$(a \vee b) \vee (a' \wedge b') = \{(a \vee b) \vee a'\} \wedge \{(a \vee b) \vee b'\}$ $= \{a \vee (b \vee a')\} \wedge \{a \vee (b \vee b')\}$ $= \{(a \vee a') \vee b\} \wedge (a \vee 1)$ $= (1 \vee b) \wedge (a \vee 1)$ $= 1 \wedge 1$ $= 1$ $(a \vee b) \wedge (a' \wedge b') = \{(a \wedge b) \wedge a'\} \vee \{(a \wedge b) \wedge b'\}$ $= \{a \wedge (b \wedge a')\} \vee \{a \wedge (b \wedge b')\}$ $= \{(a \wedge a') \wedge b\} \vee (a \wedge 1)$ $= (1 \wedge b) \vee (a \wedge 1)$ $= 0 \vee 0$ $= 0$ hence $(a \vee b)' = a' \wedge b'$ By the principle of duality, we have $(a \wedge b)' = a' \vee b'$
3(b)	**Show that direct product of any two distributive lattices is a distributive lattice.** **Proof:** Let L_1 and L_2 be two distributive lattices. Let x, y, z $\in L_1$ X L_2 be the direct product of L_1 and L_2 Then x = (a_1, a_2), y = (b_1, b_2) and z = (c_1, c_2) Now $x \vee (y \wedge z) = (a_1, a_2) \vee ((b_1, b_2) \wedge (c_1, c_2))$ $= ((a_1, a_2) \vee (b_1, b_2)) \wedge ((a_1, a_2) \vee (c_1, c_2))$ $= (x \vee y) \wedge (x \vee z)$ Thus direct product of any two distributive lattice is again a distributive lattice
4(a)	**State and prove the necessary and sufficient condition for a lattice to be modular.** **Statement:** A lattice L is modular if and only if none of its sub lattices is isomorphic to the pentagon lattice N_5 **Proof:** Since the pentagon lattice N_5 is not a modular lattice. Hence any lattice having pentagon as a sub lattice cannot be modular. Conversely, let $(L, \leq)$ be any non modular lattice and we shall prove there is a sub lattice which is isomorphic to N_5.
4(b)	**Prove that every distributive lattice is modular. Is the converse true? Justify your claim.** **Proof:** Let $(L, \leq)$ be a distributive lattice, for all a, b, c $\in L$, we have $a \oplus (b * c) = (a \oplus b) * (a \oplus c)$ Thus if $a \leq c, \quad then \quad a \oplus c = c$ $\therefore \quad a \oplus (b * c) = (a \oplus b) * c$ So if $a \leq c$, the modular equation is satisfied and L is modular. However, the converse is not true, because diamond lattice is modular but not distributive.
5(a)	**In a lattice $(L, \leq, \geq)$, prove that** $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$

	Solution: $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (a \wedge b) \vee [(b \wedge c) \vee c] \wedge [(b \wedge c) \vee a]$ $= (a \wedge b) \vee [c \wedge [(b \wedge c) \vee a]$ $= [(a \wedge b) \vee c] \wedge [(a \wedge b) \vee [(b \wedge c) \vee a]$ $= [(a \wedge b) \vee c] \wedge [(b \wedge c) \vee a]$ $= [c \vee (a \wedge b)] \wedge [a \vee (b \wedge c\}]$ $= [(c \vee a) \wedge (c \vee b)] \wedge [(a \vee b) \wedge (a \vee c)]$ $= [(c \vee a) \wedge (b \vee c)] \wedge [(a \vee b) \wedge (c \vee a)]$ $= (c \vee a) \wedge (b \vee c) \wedge (a \vee b)$ $= (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$
5(b)	**Prove that every finite lattice is bounded.** **Proof:** Let $(L, \wedge, \vee)$ be given Lattice. Since L is a lattice both GLB and LUB exist. Let "a" be GLB of L and "b" be LUB of L. For any x $\in L$, we have $a \leq x \leq b$ $GLB\{a, x\} = a \wedge x = a$ $LUB\{a, x\} = a \vee x = x$ and $GLB\{x, b\} = x \wedge b = x$ $LUB\{x, b\} = x \vee b = b$ Therefore any finite lattice is bounded.
6(a)	**In a lattice if $a \leq b \leq c$, show that** $\boldsymbol{(i)\ a \oplus b = b * c}$ $\boldsymbol{(ii)\ (a * b) \oplus (b * c) = (a \oplus b) * (a \oplus c) = b}$ **Proof:** (i) Given $a \leq b \leq c$ Since $a \leq b \Rightarrow a \oplus b = b, a * b = a \quad \ldots(1)$ $b \leq c \Rightarrow b \oplus c = c, b * c = b \quad \ldots(2)$ $a \leq c \Rightarrow a \oplus c = c, a * c = a \quad \ldots(3)$ From (1) and (2), we have $a \oplus b = b = b * c$ (ii) LHS $(a * b) \oplus (b * c) = a \oplus b = b$ RHS $(a \oplus b) * (a \oplus c) = b * c = b$ Therefore $(a * b) \oplus (b * c) = (a \oplus b) * (a \oplus c) = b$
6(b)	**In a Distributive lattice $\{L, \vee, \wedge\}$ if an element $a \in L$ is a complement then it is unique.** **Proof:** Let a be an element with two distinct complement b and c. Then a*b = 0 and a*c = 0 Hence a*b = a*c Also

	$a \oplus b = 1 \quad and \quad a \oplus c = 1$ $\therefore \ a \oplus b = a \oplus c$ Hence b = c.
7(a)	**Show that in a distributive lattice and complemented lattice $a \le b \Leftrightarrow a * b' = 0 \Leftrightarrow a' \oplus b = 1 \Leftrightarrow b' \le a'$** **Proof:** $a \le b \Leftrightarrow a * b' = 0 \Leftrightarrow a' \oplus b = 1 \Leftrightarrow b' \le a'$ **Claim 1:** $a \le b \Rightarrow a * b' = 0$ Since $a \le b \Rightarrow a \oplus b = b, a * b = a$ Now $a * b' = ((a * b) * b') = (a * b * b') = a * 0 = 0$ **Claim 2:** $a * b' = 0 \Rightarrow a' \oplus b = 1$ We have $a * b' = 0$ Taking complement on both sides, we have $(a * b')' = (0)' \Rightarrow a' \oplus b = 1$ **Claim 3:** $a' \oplus b = 1 \Rightarrow b' \le a'$ $a' \oplus b = 1 \Rightarrow (a' \oplus b) * b' = 1 * b' \Rightarrow (a' * b') \oplus (b * b') = b' \Rightarrow (a' * b') \oplus 0 = b'$ $a' * b' = b' \Rightarrow b' \le a'$ **Claim 4:** $b' \le a' \Rightarrow a \le b$ We have $b' \le a'$ taking complement we get $b' \le a' \Rightarrow a \le b$
7(b)	**In a Boolean algebra prove that $(a \wedge b)' = a' \vee b'$** **Proof:** $(a \wedge b) \vee (a' \vee b') = \{(a \wedge b) \vee a'\} \wedge \{(a \wedge b) \vee b'\}$ $= \{(a \vee a') \wedge (b \vee a')\} \vee \{(a \vee b') \wedge (b \vee b')\}$ $= \{1 \wedge (b \vee a')\} \vee \{(a \vee b') \wedge 1\}$ $= b \vee b'$ $= 1$ $(a \wedge b) \wedge (a' \vee b') = \{(a \wedge b) \wedge a'\} \wedge \{(a \wedge b) \wedge b'\}$ $= \{a \wedge a' \wedge b\} \vee \{a \wedge b \wedge b'\}$ $= \{0 \wedge b\} \vee \{a \wedge 0\}$ $= 0$ Hence proved.
8(a)	**In any Boolean algebra, show that $ab' + a'b = 0$ if and only if a = b** **Proof:** Let a = b Now $ab' + a'b = aa' + a'a = 0 + 0 = 0$ Conversely let $ab' + a'b = 0$ Now $ab' + a'b = 0 \Rightarrow ab' = -a'b = a'b$ $and \quad a = a.1 = a(b + b') = ab + ab' = ab + a'b = (a + a')b = 1.b = b$

8(b)	**Simplify** $(i)\,(a*b)' \oplus (a \oplus b)' \quad (ii)\,(a'*b'*c) \oplus (a*b'*c) \oplus (a*b'*c')$ **Solution:** $(i)\;(a*b)' \oplus (a \oplus b)' = (a \oplus b)' \oplus (a*b)'$ $= \left[(a \oplus b)' \oplus a'\right] * \left[(a \oplus b)' \oplus b'\right] = a'*b'$ $(ii)\;(a'*b'*c) \oplus (a*b'*c) \oplus (a*b'*c') = (a' \oplus a)*(b'*c) = b'*c$
9(a)	**In a Boolean algebra prove that** $(i)\,a*(a \oplus b) = a \quad (ii)\,a \oplus (a*b) = a \quad for\ all\ a, b \in B$ **Proof:** $(i)\,a*(a \oplus b) = (a+0)*(a \oplus b)$ $= a + (0*b)$ $= a + (b*0) = a + 0 = a$ Similarly by duality we have $a \oplus (a*b) = a$
9(b)	**Show that in any Boolean algebra,** $(a+b')(b+c')(c+a') = (a'+b)(b'+c)(c'+a)$ **Proof:** $(a+b')(b+c')(c+a') = (a+b'+0)(b+c'+0)(c+a'+0)$ $= (a+b'+cc')(b+c'+aa')(c+a'+bb')$ $= (a+b'+c).(a+b'+c').(b+c'+a).(b+c'+a').(c+a'+b).(c+a'+b')$ $= (a'+b+)cc'(b'+c+aa')(c'+a+bb')$ $= (a'+b+0)(b'+c+0)(c'+a+0)$ $= (a'+b)(b'+c)(c'+a)$
10(a)	**Show that in any Boolean algebra,** $a\bar{b} + b\bar{c} + c\bar{a} = \bar{a}b + \bar{b}c + \bar{c}a$ **.** **Solution:** Let (B, +, 0, 1) be any Boolean algebra and a, b, c $\in$ B. $a\bar{b} + b\bar{c} + c\bar{a} = a\bar{b}.1 + b\bar{c}.1 + c\bar{a}.1$ $= a\bar{b}(c+\bar{c}+)b\bar{c}(a+\bar{a}) + c\bar{a}(b+\bar{b})$ $= a\bar{b}c + a\bar{b}\bar{c} + ab\bar{c} + \bar{a}b\bar{c} + \bar{a}bc + \bar{a}\bar{b}c$ $= (a\bar{b}c + \bar{a}\bar{b}c) + (ab\bar{c} + a\bar{b}\bar{c}) + (\bar{a}b\bar{c} + \bar{a}bc)$ $= (a+\bar{a})\bar{b}c + (b+\bar{b})a\bar{c} + (c+\bar{c})\bar{a}b$ $= 1.\bar{b}c + 1.a\bar{c} + 1.\bar{a}b$ $= \bar{a}b + \bar{b}c + \bar{c}a$ $\therefore\; a\bar{b} + b\bar{c} + c\bar{a} = \bar{a}b + \bar{b}c + \bar{c}a$
10(b)	**Apply Demorgan's theorem to the following expression** $(i)\;\overline{(x+\bar{y})(\bar{x}+y)} \quad (ii)\;\overline{(a+b+c)d}$

Solution:

$$(i)\quad \overline{\left(x+\overline{y}\right)(\overline{x}+y)} = \overline{\left(x+\overline{y}\right)} + \overline{(\overline{x}+y)}$$

$$= \overline{x}\,.\,\overline{\overline{y}} + \overline{\overline{x}}\,.\,\overline{y}$$

$$= \overline{x}.\,y + x.\,\overline{y}$$

$$= x \oplus y$$

$$(ii)\quad \overline{(a+b+c)d} = \overline{a+b+c} + \overline{d}$$

$$= \overline{a}\,.\,\overline{b}\,.\,\overline{c} + \overline{d}$$